Math 711: Lecture of October 13, 2004

We next prove:

Theorem. *Let K be a field and let S be a finitely generated K-algebra (finite presentation is automatic). The following conditions are equivalent:*

(a) *S is smooth over K.*
(b) *$L \otimes_K S$ is smooth over L for some (equivalently, every field) L.*
(c) *For some algebraically closed field $L \supseteq K$, $L \otimes_K S$ is regular.*
(d) *For every field $L \supseteq K$, $L \otimes_K S$ is regular.*
(e) *For every maximal ideal Q of S, $(\Omega_{S/K})_Q$ is S_Q-free of rank equal to* $\dim(S_Q)$.

Proof. We shall show that (a) $\Rightarrow$ (b) for all $L \Rightarrow$ (b) for some $L \Rightarrow$ (c) $\Rightarrow$ (d) $\Rightarrow$ (e) $\Rightarrow$ (a). That (a) $\Rightarrow$ (b) for all L is immediate from our results on base change, and this evidently implies (b) for some L. If $L \otimes_K S$ is smooth over L, and $\overline{L}$ is an algebraic closure of L, then $\overline{L} \otimes_K R$ is smooth over $\overline{L}$, and the result of the previous lecture then implies that $\overline{L}_K \otimes_R S$ is regular. Thus, (b) $\Rightarrow$ (c). We have already seen that (c) and (d) are equivalent.

Now assume that (d) holds. In particular, (d) holds when L is an algebraic closure of K. Let $S' = L \otimes_K S$. Then $\Omega_{S'/L} \cong L \otimes_K \Omega_{S/K} \cong S' \otimes_S \Omega_{S/K}$. Let Q' be a maximal ideal of S' lying over a given maximal ideal Q of S. Then $\dim(S'_{Q'}) = \dim(S_Q)$ and it now suffices to see that if M is a finitely generated module over S_Q the becomes free when we tensor with $S'_{Q'}$ then it was already free. This is clear, because a minimal resolution over S_Q is preserved by applying $S'_{Q'} \otimes_{S_Q} \_$.

Finally, we need to see that (e) implies (a). This follows because (e) continues to hold after we tensor with an algebraically closed field L containing K. This implies that $L \otimes_K S$ is smooth, and therefore regular. It follows that S is regular, and, therefore, every S_Q is regular. Fix a maximal ideal Q of S and write $S = K[x_1, \ldots, x_n]_{\widetilde{Q}}/(F_1, \ldots, F_m)$, where $\widetilde{Q}$ is the inverse image of Q in a polynomial ring $K[x_1, \ldots, x_n]$ mapping onto S and $F_1, \ldots, F_m \in K[x_1, \ldots, x_n]$ are minimal generators for the kernel $K[x]_{\widetilde{Q}} \to S_Q$. We know that $(\Omega_{S/K})_Q$ is free of rank equal to $\dim(S_Q)$, and we want to show that it is free of rank $n - m$. But since S_Q is regular, we know that these two numbers are equal. $\square$

Theorem. *Let S be a finitely generated R-algebra. Then the following are equivalent:*
(a) *S is smooth over R.*
(b) *S is R-flat and every fiber $\kappa_P \otimes_R S$ is geometrically regular, where $\kappa_P = R_P/PR_P$.*
(c) *S is R-flat and for every maximal ideal Q of S lying over P in* $\mathrm{Spec}(R)$, *$(\Omega_{S/R})_Q$ is free of rank equal to* $\dim(\kappa_P \otimes_R S_Q)$.

Proof. We shall prove (a) $\Rightarrow$ (b) $\Rightarrow$ (c) $\Rightarrow$ (a). We already know that smooth algebras are flat, since they are locally polynomial followed by étale, and that fibers, which are the result of a base change, are smooth and therefore geometrically regular. Thus, (a) $\Rightarrow$ (b).

Next assume that S is R-flat, fix a maximal ideal Q, and write $S_Q \cong R[x]_{\widetilde{Q}}/IR[x]_{\widetilde{Q}}$ where $F_1, \dots, F_m \in R[x]$ are minimal generators for $IR[x]_{\widetilde{Q}}$, as usual. The sequence

$$0 \to IR[x]_{\widetilde{Q}} \to R[x]_{\widetilde{Q}} \to S_Q \to 0$$

remains exact when we apply $\kappa_P \otimes_{R_P} \_$: since S is R-flat, S_Q is R_P flat and

$$\mathrm{Tor}_1^{R_P}(\kappa_P,\, S_Q) = 0.$$

It follows that the minimum number of generators m for the kernel of $K[x]_{\widetilde{Q}} \twoheadrightarrow S_Q$ does not change when we apply $\kappa_P \otimes_{R_P}$: it is still m. Let $\mathcal{J}$ denote the image of $(\dfrac{\partial F_j}{\partial x_i})$ in S. Now, $(\Omega_{\kappa_P \otimes_R S/\kappa_P})_Q$ is free of rank $n-m$ iff $\kappa_P \to \kappa_P \otimes_{R_P} S_Q$ is formally smooth iff $\kappa_P \otimes \mathcal{J}$ has rank m if and only if a size m minor of $\mathcal{J}$ has invertible image in S_Q. It is now clear that in the presence of flatness for $R \to S$, we have that (b) $\Rightarrow$ (c) $\Rightarrow$ (a). $\square$

This completes our basic treatment of unramified, étale, and smooth morphisms. We next want to use our knowledge of étale morphisms to construct the Henselization of a quasilocal ring, as well as to gain understanding of what a Henselian ring is.

We recall that a quasilocal ring (R, P, K) is called *Henselian* if for every monic polynomial $F \in R[x]$ and each factorization $\overline{F} = gh$ into relatively prime monic polynomials g, h of $K[x]$, where $\overline{F}$ denotes the image of F in $K[x]$, there is a lifting of that factorization $F = GH$ to $R[x]$, where G and H are monic and $G \equiv g \bmod PR[x]$ while $H \equiv h \bmod PR[x]$. It is a theorem, Hensel's lemma, that if R is complete and P-adically separated then such a lifted factorization exists. That is, a complete P-adically separated ring is Henselian. In this case, and whenever R is P-adically separated, the factorization is unique: this follows from its uniqueness mod P^t for all t. See the Lecture Notes of January 9 from Math 615.

A module-finite extension ring S of a quasilocal ring (R, P, K) is said to *decompose* if it is a product of quasilocal rings. By a *pointed étale extension* of a quasilocal ring (R, P, K) we mean a localization (S, Q, L) of an étale algebra over R at a prime lying over P such that the induced map of residue fields $R/P \to S/Q$ is an isomorphism. Note that a pointed étale extension actually is an extension, since it is faithfully flat.

We shall prove:

Theorem. *Let (R, P, K) be quasilocal. Then the following seven conditions are equivalent:*

(1) *R is Henselian.*
(2) *Every module-finite extension of R decomposes.*
(3) *Every free module-finite extension of R decomposes.*
(4) *Every module-finite extension of R of the form $R[x]/(F)$, where F is a monic polynomial, decomposes.*
(5) *If F is a monic polynomial over R whose reduction $\overline{F}$ mod P has a simple root $\lambda \in K$, then there is an element $r \in R$ such that $r \equiv \lambda$ mod P and $F(r) = 0$.*
(6) *If $R \to S$ is a pointed étale extension, then $R \cong S$.*

(7) *If* $F_1, \ldots, F_n$ *are* n *polynomials in* n *variables whose images* $\overline{F}_j$ *mod* P *vanish simultaneously at* $(\lambda_1, \ldots, \lambda_n) \in K^n$ *and the Jacobian determinant* $\det\left(\frac{\partial F_j}{\partial x_i}\right)$ *does not vanish mod* P *at* $x_1 = \lambda_1, \ldots, x_n = \lambda_n$, *then there are unique elements* $r_1, \ldots, r_n \in R$ *such that for all* i, $r_i \equiv \lambda_i$ *modulo* P *and* $F_j(r_1, \ldots, r_n) = 0$, $1 \leq j \leq n$.

The proof is postponed for a bit. It is easiest to make the connection between (1) and (4), and we shall discuss this first. Note that (5) is more special than (7), and we could have made a uniqueness statement in (5) as well as in (7).