Math 711: Lecture of October 22, 2004

Let (R, P, K) be a quasilocal ring. By a *Henselization* (S, Q, L) for R we mean a quasilocal ring together with a local homomorphism $R \to S$ such that every local homomorphism from R to a Henselian quasilocal ring T, the map $R \to T$ factors uniquely $R \to S \to T$. If S' is another Henselization of R the mapping properties give that $R \to S'$ factors uniquely as $R \to S \to S'$ and $R \to S$ factors uniquely as $R \to S' \to S$. The maps $S \to S'$ and $S' \to S$ must compose to give the respective identity maps on S and S'. Thus, a Henselization of R is unique up to unique isomorphism.

We can now construct the Henselization of a quasilocal ring (R, P, K) as follows: choose a set $\mathcal{R}$ of pointed étale extensions of R containing exactly one representative of every isomorphism class. Then $\mathcal{R}$ is a directed set indexing itself, and when $S \le T$ there is a unique local R-algebra map $S \to T$. We may therefore take the direct limit, $\varinjlim_{S\in\mathcal{R}} S$. We denote this direct limit as R^{h}.

Theorem. *R^{h} is a Henselization of the quasilocal ring (R, P, K). It is faithfully flat over R, and has maximal ideal PR^{h}. Its residue class field is K.*

Proof. Let $g : R \to T$ be a local map from R to a Henselian ring T. We want to show that it has a unique local extension to R^{h}, and it suffices to show that it has a unique local extension to (S, Q, K) for every pointed étale extension S of R. Note that we have $S \otimes_R T \to K \otimes_R L \cong L$: call the kernel $\mathcal{Q}$. Then $(S \otimes_R T)_{\mathcal{Q}}$ is a localization of an étale extension of T, $T \to (S \otimes_R T)_{\mathcal{Q}}$ induces an isomorphism $L \cong K \otimes_R L \cong L$ of residue fields. Thus, $(S \otimes_R T)_{\mathcal{Q}}$ is a pointed étale extension of T. Since T is Henselian, it is equal to T. The local map $S \to (S \otimes_R T)_{\mathcal{Q}} = T$ is the map we want. The fact that $(S \otimes_R T)_{\mathcal{Q}} = T$ also implies uniqueness.

Since R^h is a direct limit of flat quasilocal rings and local R-algebra homomorphisms, it is a flat R-algebra, and the map $R \to R^{\mathrm{h}}$ is local, so that R^{h} is faithfully flat. P expands to the maximal ideal of R^{h} because that is true for every pointed étale extension, and $K \cong R^{\mathrm{h}}/PR^{\mathrm{h}}$ because $K \cong S/PS$ for every pointed étale extension S of R. □

We pause in our treatment of Henselization to consider further our results on smooth homomorphisms. The following result was stated in the lecture of September 20 as the last Theorem, to be proved later:

Theorem. *Let S be a finitely presented R-algebra.*
(a) *If R contains the rationals, S is smooth over R if and only if S is flat over R and $\Omega_{S/R}$ is projective as an R-module.*
(b) *S is étale over R if and only if S is flat over R and $\Omega_{S/R}$ is 0.*
(c) *S is unramified over R if and only if $\Omega_{S/R} = 0$.*

We eventually proved (b) and (c), but we never proved (a), although we did give criteria for smoothness involving the local freeness of the module of differentials subject to a condition on its rank.

With no condition on the rank, the hypothesis of characteristic 0 (or some other additional hypothesis) is needed. For example, let R be a field K of characteristic $p > 0$ and let $S = K[x]/(x^p)$. Then $d(x^p) = 0$, and from this is it follows that $\Omega_{S/R}$ is S-free of rank one on dx.

We want to prove (a) in equal characteristic 0. To analyze the situation it will help to consider differentials over complete local rings (R, m, K) with coefficient field K, but we want to use a somewhat different notion in this case: the ordinary module of differentials $\Omega_{R/K}$ need not be finitely generated, but its m-adic completion $\widehat{\Omega}_{R/K}$ is. (The power series ring $R = K[[x_1, \ldots, x_n]]$ typically has uncountable transcendence degree over K, even when $n = 1$. If K has characteristic 0, and $\{u_j\}_{j\in J}$ is a transcendence basis in R for $\mathcal{F}$, the fraction field of R, over K, then because the extension $K(u_j : j \in J) \subseteq \mathcal{F}$ is separable, we have that $\Omega_{\mathcal{F}/K}$ is free on the du_j, and needs uncountably many generators. This is a localization of $\Omega_{T/K}$, which must also need uncountably many generators. In characteristic p, if K is perfect then any derivation kills $T_0 = K[[x_1^p, \ldots, x_n^p]]$. Since T is module-finite over T_0 it is certainly finitely generated as an algebra over T_0 by the elements dx_j. It follows that $\Omega_{T/K} = \Omega_{T/T_0}$ is a finitely generated T-module and so is already complete and $\widehat{\Omega}_{T/K} = \Omega_{T/K}$ in this case.)

In order to see that $\widehat{\Omega}_{R/K}$ is finitely generated, we first want to define $\dfrac{\partial}{\partial x}$ for a power series ring $R[[x]]$. We may define this formally by the rule

$$\frac{\partial}{\partial x}\sum_{i=0}^{\infty} r_i x^i = \sum_{i=1}^{\infty} r_i i x^{i-1},$$

where the $r_i \in R$. Alternatively, we may obtain $\dfrac{\partial}{\partial x}F$ by introducing a new formal indeterminate Δ, noting that $F(x+\Delta) - F(x)$ is divisible by Δ in $R[[x, \Delta]]$, and letting

$$\frac{\partial}{\partial x}F = \frac{F(x+\Delta) - F(x)}{\Delta}\bigg|_{\Delta=0}.$$

This is an R-derivation of $R[[x]]$ into itself. Now, consider $A[[x_1, \ldots, x_n]]$, the formal power series ring in n variables over A. We may define $\dfrac{\partial}{\partial x_i}$ for $1 \leq i \leq n$ by letting R_i be the formal power series ring over A in the variables other than x_i and thinking of $A[[x_1, \ldots, x_n]]$ as $R_i[[x_i]]$.

Now let (R, m, K) be a complete local ring with coefficient field K, so that we have $K \hookrightarrow R$ as well as $R \twoheadrightarrow K$ and the composition is the identity. We write $\widehat{\Omega}_{R/K}$ for the m-adic completion of $\Omega_{R/K}$. By a *complete* R-module M we mean an R-module that is complete and separated in the m-adic topology. The composition $d : R \to \Omega_{R/K} \to \widehat{\Omega}_{R/K}$ gives a K-derivation of R into $\widehat{\Omega}_{R/K}$ which we still denote by d.

Then:

Proposition. *Let (R, m, K) be complete local with coefficient field K.*

(a) *For every complete R-module M, there is a bijection between $\mathrm{Hom}_R(\widehat{\Omega}_{R/K}, M)$ and K-derivations of R into M: every derivation is obtained from a unique R-linear map $L : \widehat{\Omega}_{R/K} \to M$ by composition with d. This mapping property together with the condition that $\widehat{\Omega}_{R/K}$ be complete characterizes $\widehat{\Omega}_{R/K}$ up to unique isomorphism.*

(b) *If R is a formal power series ring $K[[x_1, \dots, x_n]]$, then $\widehat{\Omega}_{R/K}$ is the free R-module on the basis dx_i, and*

$$dF = \sum_i \frac{\partial F}{\partial x_i} dx_i$$

(c) *If R is the quotient of $K[[x_1, \dots, x_n]]$ by the ideal with generators $F_1, \dots, F_m$, $\widehat{\Omega}_{R/K}$ is quotient of the free R-module on the dx_i by the images of the df_j: thus $\widehat{\Omega}_{R/K}$ is the cokernel of the Jacobian matrix $(\dfrac{\partial F_j}{\partial x_i})$, which is consequently independent of the choice of presentation $K[[x_1, \dots, x_n]]/(F_1, \dots, F_m)$. In particular, $\widehat{\Omega}_{R/K}$ is finitely generated.*

Proof. For part (a), the derivation induces a unique linear map $\Omega_{R/K} \to M$: since M is complete, this factors uniquely through $\widehat{\Omega}_{R/K}$. The rest of the argument is routine.

For part (b), it suffices to see that a K-derivation D of R into a complete module M is uniquely determined by the images of the dx_i, and that there is a derivation for every specified set of values. If values $u_1, \dots, u_n \in M$ are specified one simply checks that the map taking $F \in K[[x_1, \dots, x_n]]$ to $\sum_{i=1}^n \frac{\partial F}{\partial x_i} u_i$ is a derivation. To see that the derivation is determined by its values on the x_i, fix $N \in \mathbb{N}$. Given F, let f be the polynomial containing all terms of F of degree $\leq 2N$, and let $w \in P^{2N}$ be the sum of remaining terms of F. The definition of a derivation forces

$$dF = \sum_{i=1}^n \frac{\partial f}{\partial x_i} u_i$$

and since $w = F - f \in P^{2N} = (P^N)^2$, the product rule forces Dw to be in $P^N M$. Thus,

$$DF - \sum_i \frac{\partial F}{\partial x_i} u_i \in P^N M$$

for all N, and the result follows.

To prove part (c), we note that a K-derivation $R = K[[x_1, \dots, x_n]]/(F_1, \dots, F_m) \to M$ induces a K-derivation $K[[x_1, \dots, x_n]] \twoheadrightarrow K[[x_1, \dots, x_n]]/(F_1, \dots, F_m) \to M$ by composition. The condition that a K-derivation $K[[x_1, \dots, x_n]] \to M$ factor through $K[[x_1, \dots, x_n]]/(F_1, \dots, F_m)$ is simply that every dF_j be mapped to 0, and so the cokernel over R of $(\dfrac{\partial F_j}{\partial x_i})$ has the required mapping property. $\square$

Proposition. *Let (R, P, K) be complete with coefficient field $K \subseteq R$.*

(a) *If K has characteristic 0, then R is regular if and only if $\widehat{\Omega}_{R/K}$ is free, in which case it is free of rank* $\dim(R)$.

(b) *If K has characteristic $p > 0$, then R is regular if and only of $\widehat{\Omega}_{R/K}$ is free of rank $\le \dim(R)$, in which case it is free of rank $\dim(R)$. If K is perfect, R is reduced, and $\widehat{\Omega}_{R/K}$ is free, then R is regular.*

Proof. We can represent R as a quotient $K[[x_1, \dots, x_n]]/(F_1, \dots, F_m)$ in such a way that $F_1, \dots, F_m$ are minimal generators of I and all $F_j \in (x_1, \dots, x_n)^2$: if any F_j has a nonzero linear form, $K[[x_1, \dots, x_n]]/(F_j)$ is regular and can be written as power series ring in fewer variables. Then $\widehat{\Omega}_{R/K}$ is the cokernel of the image of $\left(\frac{\partial F_j}{\partial x_i}\right)$, which has entries in P, and so the cokernel is free if and only if all of the partial derivatives are in the ideal. In characteristic 0 this cannot happen unless all the F_j are 0, by the Lemma that follows. In characteristic p, again by the Lemma that follows, one has that every variable occurs in every term of every F_j with exponent that is a multiple of p. Therefore, all the F_j are pth powers if K is perfect. In this case, the module of differentials is free of rank n, but if some F_j is nonzero then $\dim(R) < n$. If K is perfect the fact that R is reduced and F_j is a pth power contradicts the assumption that F_j is a minimal generator of m. $\square$