Math 711: Lecture of October 25, 2004

Lemma. *Let K be a field and let $T = K[[x_1, \ldots, x_n]]$ be a formal power series ring over K. Let $I = (f_1, \ldots, f_m)$ be an ideal, and suppose that for all i, j, $\dfrac{\partial f_j}{\partial x_i} \in I$.*

(a) *If K has characteristic 0, then $I = 0$ or $I = T$.*
(b) *If K has characteristic $p > 0$, then I is generated by elements of $K[[x_1^p, \ldots, x_n^p]]$. If K is perfect, these elements are pth powers.*

Proof. First note that if all the partial derivatives of all of a given set of generators of I are in I, then I is closed under partial differentiation:

$$\frac{\partial}{\partial x_i}\sum_j g_j f_j = \sum_j \Big(\frac{\partial g_j}{\partial x_i} f_j + g_j \frac{\partial f_j}{\partial x_i}\Big),$$

and all terms are in I.

In the characteristic 0 case, if I is not (0), choose an element h of I whose lowest degree term H is of smallest degree. If $I \neq R$, then H has positive degree. Choose a variable x_i that occurs in H. Then $\dfrac{\partial H}{\partial x_i} \neq 0$ and is the lowest degree term of $\dfrac{\partial h}{\partial x_i}$, a contradiction.

In the case of characteristic $p > 0$, note that the set $\mathcal{M}$ of monomials $\mu = x_1^{a_1} \cdots x_n^{a_n}$ such that all of the $a_t < p$ form a free basis for T over $T_0 = K[[x_1^p, \ldots, x_n^p]]$. Also note that all of the derivations $\dfrac{\partial}{\partial x_i} : T \to T$ are T_0-linear. Suppose that I has an element $\sum_{\mu \in \mathcal{M}} g_\mu \mu$ where the $g_\mu \in T_0$. We want to show that all of the $g_\mu \in I$. Evidently, if we have a counterexample, we still have a counterexample if we omit all terms such that $g_\mu \in I$. Therefore we may assume that $g_\mu \notin I$ for all μ that occur, i.e., such that $g_\mu \neq 0$. Choose $\mu^* = x_1^{a_1} \cdots x_n^{a_n}$ occurring of highest degree. Then apply

$$D = \frac{\partial^{a_1}}{\partial x_1^{a_1}} \cdots \frac{\partial^{a_n}}{\partial x_n^{a_n}}.$$

The value on μ^* is $a_1! \cdots a_n!$. All other μ occurring are killed by D, since some exponent in μ will be strictly less than the corresponding exponent a_i in μ^*, and D is T_0-linear. Thus, $a_1! \cdots a_n! g_{\mu^*} \in I$, and it follows that $g_{\mu^*} \in I$, a contradiction. $\square$

Corollary. *Let S be finitely presented and R-flat. If R contains the rational numbers, S is smooth over R iff $\Omega_{S/R}$ is a locally free S-module. More generally, S is smooth over R if and only if for every maximal ideal Q of S lying over P in R, $(\Omega_{S/R})_Q$ is free of rank less than or equal to the dimension of the localized fiber S_Q/PS_Q, in which case its rank is that dimension.*

Proof. We have already proved the necessity of these conditions. To see sufficiency, note that it suffices to show that the localized fibers at maximal ideals are formally smooth.

Thus, we may assume that R is a field K. We may make a base change to the algebraic closure of K without affecting the issue. Therefore, we may assume that S_Q has residue field K, and K is a coefficient field. The regularity of S_Q and the freeness and rank of Ω are unaffected by completion if we use $\widehat{\Omega}_{R/K}$ in the complete case. The result is now immediate from the preceding Proposition. $\square$

Remark. When S is finitely presented and flat over R, it is smooth if and only if for every maximal ideal Q of S with contraction P to R, S_Q/PS_Q is formally smooth over $K = R_P/PR_P$. Therefore, suppose that we restrict attention to the case where $R = K$ is a field. We may make a base change to the case where K is algebraically closed without affecting smoothness. Suppose that K is a perfect field of positive characteristic. Then S_Q is formally smooth over K if and only if $(\Omega_{S/K})_Q$ is S_Q-free and S_Q is reduced. However, the proof needs one fact that we have not established: if a local ring of a finitely generated K-algebra is reduced, then so is its completion.

We return to the subject of Henselization.

Proposition. *If I is a proper ideal of a quasilocal ring R, then there is a canonical local R-isomorphism $(R/I)^{\mathrm{h}} \cong R^h/IR^h$ (since both are killed by I, this is equivalent to saying that there is a canonical local (R/I)-isomorphism).*

Proof. A homomorphic image of a Henselian ring is Henselian: given the problem of lifting a factorization over the residue class field K, one can lift to R, and then take the image of that lifting in the quotient. Thus, $R^{\mathrm{h}}/IR^{\mathrm{h}}$ is Henselian, and so $R/I \to R^{\mathrm{h}}/IR^{\mathrm{h}}$ induces a unique (R/I)-algebra map $(R/I)^{\mathrm{h}} \to R^{\mathrm{h}}/IR^{\mathrm{h}}$. Likewise, the map $R \to (R/I)^{\mathrm{h}}$ induces a unique local R-algebra map $R^{\mathrm{h}} \to (R/I)^{\mathrm{h}}$, and, since it kills I, we get a unique local (R/I)-algebra map $R^{\mathrm{h}}/IR^{\mathrm{h}} \to (R/I)^{\mathrm{h}}$. The composite map $(R/I)^{\mathrm{h}} \to (R/I)^h$ is the identity on elements of R/I and so is the identity. Consider the composite map $\alpha : R^{\mathrm{h}}/IR^{\mathrm{h}} \to R^{\mathrm{h}}/IR^{\mathrm{h}}$ and compose with the surjection $R^{\mathrm{h}} \to R^{\mathrm{h}}/IR^{\mathrm{h}}$: the resulting map $\beta : R^{\mathrm{h}} \to R^{\mathrm{h}}/IR^{\mathrm{h}}$ takes the image of $r \in R$ to its image in R^h/IR^h, and since β extends uniquely to R^{h}, it must be the quotient surjection. This means that α must be the identity map. $\square$

Theorem. *Let (R, P, K) be a Noetherian local ring. Then there are unique local maps $R \to R^{\mathrm{h}} \to \widehat{R}$ and these are injective. If S is any pointed étale extension of R this also factors uniquely $R \to S \to R^{\mathrm{h}} \to \widehat{R}$ and if we tensor with R/P^n these maps all become isomorphisms:*

$$R/P^nR \cong S/P^nS \cong R^{\mathrm{h}}/P^nR^{\mathrm{h}} \cong \widehat{R}/P^n\widehat{R}.$$

R^{h} may be canonically identified with a subring of $\widehat{R}$ and is Noetherian. Moreover, the induced maps $\widehat{R} \to \widehat{S} \to \widehat{R^{\mathrm{h}}} \to \widehat{R}$ are all isomorphisms, i.e.,

$$\widehat{R} \cong \widehat{S} \cong \widehat{R^{\mathrm{h}}} \cong \widehat{R}.$$

Proof. By Hensel's lemma, $\widehat{R}$ is Henselian, and this gives a unique local R-algebra factorization $R \to R^{\mathrm{h}} \to \widehat{R}$. Moreover, S is part of the direct limit system used to obtain R^{h}. Once we kill P^n we have that R/P^n is complete, and therefore Henselian. S/P^nS

is a pointed étale extension of R/P^n and therefore equal to it, while, $R^{\mathrm{h}}/P^n R^{\mathrm{h}}$ is the Henselization of R/P^n and so equal to R/P^n as well. This shows that for each pointed étale extension S of R, $\widehat{S} \to \widehat{R}$ is an isomorphism. If $u \in R^{\mathrm{h}}$ mapped to 0 in $\widehat{R}$ we can choose $u \in S$ for some S, and now we have a contradiction since $\widehat{S} \to \widehat{R}$ is an isomorphism. Thus, R^{h} injects into $\widehat{R}$.

Suppose that $I \subseteq R^{\mathrm{h}}$ is not finitely generated. Then we can find a sequence of finitely generated subideals $\{I_t\}_t$ that is strictly ascending. The expansions to $\widehat{R}$ must stabilize. Suppose that s is so large that $I_t\widehat{R} = I_s\widehat{R}$ for all $t \geq s$, and that $f_1, \ldots, f_n \in R^{\mathrm{h}}$ generate $I_s\widehat{R}$. We claim that these elements generate I_t for all $t \geq s$. To see this choose $g \in I_t$. Choose a pointed étale extension S of R sufficiently large that it contains all of $f_1, \ldots, f_h$ and g. But after completion the isomorphisms $S/P^nS \to \widehat{R}/P^n\widehat{R}$ show that $\widehat{S} \cong \widehat{R}$, and this implies that $\widehat{R}$ is faithfully flat over S. Since $g \in (f_1, \ldots, f_h)\widehat{R}$, we must have $g \in (f_1, \ldots, f_h)S \subseteq (f_1, \ldots, f_h)R^{\mathrm{h}}$. The final statement is clear from the isomorphisms mod every P^n. $\square$