Math 711: Lecture of October 27, 2004

If R is Noetherian local, one may take a representative of every isomorphism class of pointed étale extensions inside $\widehat{R}$, and the directed union of these is a canonical Henselization of R. If (R, P, K) is any quasilocal ring, for each subalgebra B of R finitely generated over the prime ring in R, we may form $B_{P\cap B}$, which has a local map $B_{P\cap B} \to R$. Then

$$R = \varinjlim_B B_{P\cap B}$$

and

$$R^{\mathrm{h}} = \varinjlim_B \bigl(B_{P\cap B}\bigr)^{\mathrm{h}}$$

gives a canonical Henselization of B.

The ring of germs of real-valued C^∞ functions or real analytic functions at a point of $\mathbb{R}^d$ or of germs of complex analytic functions at a point of $\mathbb{C}^d$ is Henselian. The germs of real analytic functions may be identified with the subring of $\mathbb{R}[[x_1, \ldots, x_d]]$ consisting of power series that converge on some neighborhood of the origin, and a similar comment applies to complex analytic functions. The Henselian property follows because there is an implicit function theorem that applies in each case. Shift coordinates so that the point is at the origin. The map from the local ring to the residue class field consists of evaluation of a representative of the germ at the origin. Given n polynomial equations in n unknowns whose coefficients are germs, each germ may be represented by an actual function on some open neighborhood of the origin. We may restrict to the intersection of these open neighborhoods. We can now work with polynomials whose coefficients are functions. The hypothesis on the Jacobian determinant enables one to apply an implicit function theorem for the appropriate category to get functions that satisfy the equations.

One version of an implicit function theorem is this: let X, Y denote the $n+d$ variables $X_1, \ldots, X_n, Y_1, \ldots, Y_d$ and let $F_1, \ldots, F_n$ be n $\mathbb{R}$-valued C^∞ functions defined on a neighborhood of a point $(x, y) \in \mathbb{R}^{n+d}$ that vanish at that point. Suppose that $\det\bigl(\dfrac{\partial F_j}{\partial X_i}\bigr)$ (note that the matrix is $n \times n$) does not vanish at (x, y). Then there are unique C^∞ functions $g_1(Y), \ldots, g_n(Y)$ defined on a neighborhood U of y such that

$$x_j = g_j(y),\ 1 \le j \le n, \text{ and } F_j(g_1(Y), \ldots, g_n(Y), Y_1, \ldots, Y_d) = 0$$

identically on U, $1 \le j \le m$. Thus, the equations determine a unique solution for the X_i in terms of the Y_k near the point (x, y). The condition in the hypothesis is very natural if one thinks of the case where all the F_j are linear. The same statement holds if one replaces the condition that functions in both the hypothesis and conclusion be C^∞ by the condition that they be real analytic, and also if one works with $\mathbb{C}$-valued functions on $\mathbb{C}^{n+d}$ and replaces the C^∞ condition by the condition that the functions in both the hypothesis and conclusion be holomorphic.

We next want to review some material concerning excellent rings. We shall not give proofs, but we will not be making use of the theorems we do not prove. A Noetherian ring is called *catenary* if whenever $P \subseteq Q$ are prime ideals all saturated chains of primes from P to Q have the same length. This property obviously passes to quotient rings and localizations. A Noetherian ring is called *universally catenary* if every finitely generated algebra over it is catenary. It suffices that polynomial rings in finitely many variables over it be catenary: other finitely generated algebras over it are quotients of these. The Appendices to Nagata's book on local rings contain examples of Noetherian rings that are not catenary, and catenary rings that are not universally catenary. However, Cohen-Macaulay rings are catenary. Since a polynomial ring over a Cohen-Macaulay ring is again Cohen-Macaulay, we have that Cohen-Macaulay rings are universally catenary, and that means that homomorphic images of Cohen-Macaulay rings are universally catenary.

A map of rings is said to be a $\mathcal{P}$ map, where $\mathcal{P}$ is a property of rings, if it is flat and all the fibers have property $\mathcal{P}$. Thus, a map is Cohen-Macaulay if it is flat with Cohen-Macaulay fibers, and a map is geometrically regular if it is flat with geometrically regular fibers. With this terminology, $R \to S$ is smooth if and only if it is finitely presented and geometrically regular. However, some authors use the term "regular" to mean geometrically regular. In this course we shall use the term geometrically regular, and avoid the term "regular."

A Noetherian ring R is called a *G-ring* ("G" as in "Grothendieck") if for every local ring A of R, the map $A \to \widehat{A}$ is geometrically regular.

An *excellent* ring is a universally catenary Noetherian G-ring R such that in every finitely generated R-algebra S, the regular locus $\{P \in \mathrm{Spec}\,(S) : S_P \text{ is regular}\}$ is Zariski open. Excellent rings include the integers, fields, complete local rings, convergent power series rings, and are closed under taking quotients, localization, and formation of finitely generated algebras. All of the rings that come up in algebraic geometry, number theory, and several complex variables are excellent. Excellent rings tend very strongly to share the good behavior exhibited by rings that are finitely generated over a field. The normalization of an excellent domain is a finite module over it, the completion of a reduced excellent local ring is reduced, and the completion of a normal excellent local domain is normal. It is also true that the Henselization of an excellent local ring is again excellent.

In the sequel we shall refer from time to time to excellent discrete valuation rings. We always mean Noetherian rank one discrete valuation domains. We use the abbreviation DVR. In this case one can give a simpler characterization of excellence, and we shall take this characterization as our working definition of excellence for DVRs throughout the remainder of these Lecture Notes.

We first define an algebra S over a field K to be *separable* if for every finite purely separable algebraic extension L of K, the ring $L \otimes_K S$ is reduced. We shall say a bit more about this shortly. Notice that if K has characteristic 0, then every reduced K-algebra is separable. We shall say a DVR V with fraction field K is excellent if $K \otimes_V \widehat{V}$ is separable. This is equivalent to the general definition of excellence given earlier in the special case of a DVR. Notice that every DVR of equal characteristic 0 or of mixed characteristic is excellent. There can be no problem unless the ring has positive characteristic p.

We define a Noetherian ring (R, P, K) to be an *approximation ring* if every finite system of polynomial equations in finitely many variables over R that has a solution in $\widehat{R}$ has a solution in R. The reason for the use of the term "approximation" is explained in part by the following fact.

Proposition. *Let R be an approximation ring, let $F_j(X_1, \ldots, X_n) = 0$, $1 \le j \le m$ be a system of polynomial equations over R, and let $(s_1, \ldots, s_n)$ be a solution of the equations in $\widehat{R}^n$. Then for every positive integer N there exists a solution $(r_1, \ldots, r_n)$ in R^n such that for all i, $r_i \cong s_i$ mod $P^N\widehat{R}$. In other words, the solutions of the equations over R are P-adically dense in the solutions over $\widehat{R}$.*

Proof. Fix N and fix elements $r'_i \in R$ such that $r'_i \cong s_i \bmod P^N\widehat{R}$, $1 \le i \le n$. Fix generators $u_1, \ldots, u_k$ for P^n in R. For each i we can find elements $y_{ij} \in \widehat{R}$ such that $s_i - r'_i - \sum_{j=1}^k y_{ij}u_j = 0$. Now consider the equations $F_1 = 0, \ldots, F_m = 0$ together with the additional equations involving new variables Y_{ij}, namely $X_i - r'_i - \sum_{j=1}^k Y_{ij}u_j = 0$. These equations have a solution $(s_1, \ldots, s_n, y_{ij})$ in $\widehat{R}$. Hence, they have a solution in R, and for this solution the values for the X_i will satisfy the congruence condition. $\square$

Thus, solutions over $\widehat{R}$ can be "approximated" arbitrarily closely, in the P-adic topology, by solutions over R.

Based on a wonderful result of Popescu, one can prove that every excellent Henselian ring is an approximation ring. The original proof of the result of Popescu has gaps. Popescu's theorem asserts that if S is a geometrically regular R-algebra, and one has a finitely generated R-subalgebra S_0 of S, then there is a smooth R-algebra T and a factorization $R \to S_0 \to T \to S$. (It is not known that the map $T \to S$ can be taken to be injective.) This means that S is a sort of limit of smooth algebras. A proof by Ogoma filled some of the gaps in Popescu's argument, while Swan eventually wrote an exposition of the proof that definitively answered all questions about the earlier versions. The proof is very long and difficult, and we shall not prove the full result here.

We will however, prove the following beautiful result of Mike Artin, which motivated the later work.

Theorem (Artin approximation). *Let R be a local ring of a finitely generated algebra over a field or over an excellent DVR. Then R^{h} is an approximation ring. In particular, every finite system of polynomial equations over R that has a solution in $\widehat{R}$ has a solution in a pointed étale extension of R.*

This is a hard theorem: it will take us a while before we can prove it.

Artin also proved that the ring of convergent power series over $\mathbb{C}$ is an approximation ring. This is an amazing statement: it says that if a system of polynomial equations with convergent power series coefficients has a solution in the formal power series ring, then it has a solution in the convergent power series ring.

Both of these results of Artin are special cases of the statement that every excellent Henselian ring is an approximation ring.