Math 711: Lecture of October 29, 2004

We discuss separable algebras a bit further. Recall that a K-algebra R is *separable* if for every finite purely inseparable extension L of K, $L \otimes_K R$ is reduced.

Lemma. *Let K be a field and let R be a K-algebra. Let L denote an extension field of K.*
(a) *If K has characteristic 0, the R is separable if and only if it is reduced.*
(b) *If $L \otimes_K R$ is reduced, then for every subfield L_0 of L containing K, $L_0 \otimes_K R$ is reduced.*
(c) *If R is separable over K then every K-subalgebra of R is separable over K.*
(d) *A direct limit of separable K-algebras is separable.*
(e) *R is separable over K if and only if all of its finitely generated K-subalgebras are separable over K.*
(f) *If R is reduced and L is separable algebraic over K, then $L \otimes_K R$ is reduced.*
(g) *If R is reduced and L is pure transcendental over K, then $L \otimes_K R$ is reduced.*

Proof. Part (a) is immediate from the definition, since the only purely inseparable extension of K is K itself. Part (b) holds because $L_0 \otimes_K R \subseteq L \otimes_K R$, since R is K-flat. For the same reason, if R_0 is a K-subalgebra of R we have that $L \otimes_K R_0 \subseteq L \otimes_K R$, from which (c) follows. Part (d) is a consequence of the fact that tensor product commutes with direct limit, since a direct limit of reduced rings is reduced. Part (e) is immediate from (c) and (d). To prove (f), note that since L is a direct limit of finite separable algebraic extensions, we may assume that L is finite separable algebraic over K. Since R is a direct limit of finitely generated K-subalgebras, we may assume that R is reduced and finitely generated over K. Then R embeds in its localization $W^{-1}R$ at the multiplicative system of all nonzerodivisors, and because R is Noetherian and reduced, this is a finite product of fields. Thus, it suffices to prove the result when L is finite separable algebraic over K and R is a field. But then L is étale over K and so $L \otimes_K R$ is étale over R, and, consequently, a finite product of separable field extensions of R. Thus, it is reduced. Part (g) is clear because the pure transcendental extension is a localization of a polynomial ring over R. $\square$

Proposition. *Let K be a field and let R be a K-algebra. The following conditions are equivalent:*
(1) *R is separable over K.*
(2) *If L is the perfect closure of K, then $L \otimes_K R$ is reduced.*
(3) *If L is the algebraic closure of K, then $L \otimes_K R$ is reduced*
(4) *For every field extension L of K, $L \otimes_K R$ is reduced.*
(5) *For some perfect field L containing K, $L \otimes_K R$ is reduced.*

Proof. (1) and (2) are equivalent because the perfect closure of K is a directed union of finite purely inseparable extensions of K, and contains all of them. Clearly, (4) $\Rightarrow$ (3) $\Rightarrow$ (5) $\Rightarrow$ (2) (since the algebraic closure is perfect and since every perfect field containing K contains a K-isomorphic copy of the perfect closure), and so it will suffice to prove that (2) $\Rightarrow$ (4). Since every field is contained in a larger field that contains the perfect closure L of K, it suffices to show that $L' \otimes_K R$ is reduced when L' is a field containing L. L' is the directed union of finitely generated extensions L'_0 of L within L'. Therefore, we may assume that L' is finitely generated over the perfect field L. But then L' has a separating transcendence basis over L, and may be reached by a pure transcendental extension followed by a separable algebraic extension. The result now follows from parts (f) and (g) of the Lemma. $\square$

We also note:

Proposition. *If L is an algebraic field extension of K, then L is separable as a K-algebra if and only if it is a separable field extension of K.*

Proof. If L' is any field extension of K, then $L' \otimes_K L$ is reduced by part (f) of the Lemma. Now suppose that L is not a separable field extension of K. Then L contains an inseparable element θ whose minimal monic polynomial $f = f(x)$ has multiple roots. Let $\overline{K}$ denote an algebraic closure of K. Then $\overline{K} \otimes_K L \supseteq \overline{K} \otimes_K K[\theta] \cong \overline{K} \otimes_K K[x]/(f) \cong \overline{K}[x]/(f)$, and since f is not square-free over $\overline{K}$, the last ring has nilpotents. $\square$

Let k be any perfect field of positive characteristic p, and let $t_1, \dots, t_n, \dots$ be countably many indeterminates over K. Let $K_0 = K(t_i^p : i)$, and let $K_n = K_0(t_1, \dots, t_n)$, so that $K_0 \subseteq K_1 \subseteq \cdots$ is a strictly increasing sequence of fields. Let K denote the union. Let $V = \bigcup_{n=0}^{\infty} K_n[[x]]$. Every $K_n[[x]]$ is complete and, therefore, Henselian, and a direct limit of Henselian rings is Henselian. Therefore, V is Henselian. Every nonzero element of V is a unit times a power of x, since that is true in every V_n. It follows that V is a discrete valuation ring in which x generates the maximal ideal. $V/x^N \cong K[[x]]/(x^N)$, and it follows that $\widehat{V}$ may be identified with $K[[x]]$. But $K[[x]]^p = K^p[[x^p]] = K_0[[x^p]] \subseteq K_0[[x]] \subseteq V$, so that $\widehat{V}$ is purely inseparable over V, and this remains so when we localize at $V - \{0\}$ (equivalently, at x). Note that, for example, $u = \sum_{n=1}^{\infty} t_n x^n \in \widehat{V} - V$. Thus, V is a DVR that is not excellent.

Also note that Artin approximation fails for V. The equation $z^p - u^p = 0$ (note that $u^p \in V$) has the unique solution $z = u$ in the domain $\widehat{V}$, but it has no solution in V: the only possibility is $z = u$, and $u \notin V$.

It is also worth noting that approximation rings must be Henselian. Given an approximation ring (R, P, K) and a factorization of a monic polynomial over R into relatively prime factors in $K[x]$, the problem of lifting the factorization may be translated into solving a system of equations for the coefficients. It will have a solution in $\widehat{R}$, since that ring is complete, and therefore Henselian. Hence, it must have a solution in R as well.

Here is an alternative characterization of approximation rings:

Proposition. *Let R be a local ring. Then R is an approximation ring if and only if it is an R-algebra retract of every finitely generated R-subalgebra S of $\widehat{R}$, i.e., if and only if for every such S there is an R-algebra homomorphism $\rho : S \to R$ such that if $\iota : R \hookrightarrow S$ is the inclusion map, $\rho \circ \iota = \mathbf{1}_R$.*

Proof. Suppose that R is an approximation ring and let $S = R[s_1, \dots, s_n]$ be given. Map a polynomial ring $R[X_1, \dots, X_n] \twoheadrightarrow R[s_1, \dots, s_n]$ as R-algebras by sending $X_j \mapsto s_j$, $1 \le j \le n$, and let $F_1, \dots, F_m$ generate the kernel. Then $(s_1, \dots, s_n)$ is a solution for the m equations $F_j = 0$ in R. Therefore the equations have a solution $(r_1, \dots, r_n)$ in R.

The R-algebra map $R[X_1, \dots, X_n] \to R$ sending $X_j \mapsto r_j$ kills the F_j and so induces the required retraction $S \cong K[x_1, \dots, x_n]/(F_1, \dots, F_m)$ to R.

To prove the "if" part let $F_j = 0$, $1 \le j \le m$ be a system of polynomial equations over R with a solution $(s_1, \dots, s_n)$ in $\widehat{R}$, and let $S = R[s_1, \dots, s_n]$. Choose an algebra retraction ρ of S to R. Let $r_i = \rho(s_i)$, $1 \le i \le n$. Since the s_i satisfy the equations $F_j = 0$ and ρ is an R-algebra homomorphism, the r_i also give a solution. $\square$

We next want to use Zariski's Main Theorem and our theory of Henselization and étale maps to compare a very geometric notion of intersection multiplicity that we shall introduce with Serre's definition using alternating sums of lengths of Tor.

The first step is to introduce the geometric notion. To this end, we consider two algebraic varieties X, Y in $A^n_{\mathbb{C}} = \mathbb{C}^n$. That is, X and Y are irreducible closed algebraic sets. We deal only with closed points here. Suppose that u is an isolated point of intersection of $X \cap Y$. In the situation where $\dim(X) + \dim(Y) = n$ (X and Y are said to be intersecting *properly* at u, we want to introduce a positive integer that represents the intersection multiplicity of X and Y at u.

Here is one example. Suppose that $n = 2$ with coordinates x, y in $\mathbb{A}^2_{\mathbb{C}}$ and that $X = V(y - x^2)$ while $Y = V(y)$. The origin $(0, 0)$ is an isolated point of intersection of these two varieties. However if we replace the line $y = 0$ with the line $y = \epsilon$ for any small $\epsilon \neq 0$, $X = V(y - x^2)$ and $Y_\epsilon = V(y - \epsilon)$ have two points of intersection: $\epsilon \neq 0$ has two distinct square roots, and the points are $(\pm\sqrt{\epsilon}, \epsilon)$. It is therefore natural to define the intersection multiplicity to be 2 in this case.

If X and Y are arbitrary varieties one can simplify the problem of understanding the intersection by the process of "reduction to the diagonal." Let Δ denote the diagonal subvariety of $\mathbb{A}^n_{\mathbb{C}} \times \mathbb{A}^n_{\mathbb{C}} \cong \mathbb{A}^{2n}_{\mathbb{C}}$: $\Delta = \{(v, v) : v \in \mathbb{A}^n_{\mathbb{C}}\}$. This variety is a vector subspace defined by n linear equations. Set-theoretically there is a bijection between $X \cap Y$ and $(X \times Y) \cap \Delta$ under which u corresponds to (u, u). This bijection is an isomorphism of algebraic sets. Moreover, a great deal of experience has shown that one can replace the problem of understanding the manner in which X meets Y by the problem of understanding the manner in which $X \times Y$ meets Δ. For example, u is an isolated point of intersection of X and Y iff (u, u) is an isolated point of intersection of $X \times Y$ and Δ. Moreover, X and Y meet properly in $\mathbb{A}^n_{\mathbb{C}}$ iff $X \times Y$ and Δ meet properly in $\mathbb{A}^{2n}_{\mathbb{C}}$: the dimension of $X \times Y$ is $\dim(X) + \dim(Y)$.

We shall therefore focus attention on the problem of defining intersection multiplicities when one is intersecting a variety with a linear space. Note that if the linear space is defined by independent linear equations $L_1, \dots, L_d$ then one can imitate the example with the parabola by counting points of intersection with $V(L_1 - \epsilon_1, \dots, L_d - \epsilon_d)$, where the ϵ_j are small, varying complex numbers. Here, one hopes that the number of points of intersection that are "near" the isolated point u will be constant for "almost all" choices of $(\epsilon_1, \dots, \epsilon_d)$ consisting of "sufficiently small" complex values. In the case of the parabola we only had to exclude the value 0 for ϵ. Our next task is to make all of this precise.