Math 711: Lecture of November 1, 2004

Given a finitely generated $\mathbb{C}$-algebra R, if the module differentials $\Omega_{R/\mathbb{C}}$ is locally free over R then R is regular (which is equivalent to being smooth over $\mathbb{C}$). What if one assumes instead that the R-module of derivations $\mathrm{Der}_{\mathbb{C}}(R,\, R)$ is locally free? Must R be regular? Note that $\mathrm{Der}_{\mathbb{C}}(R,\, R) \cong \mathrm{Hom}_R(\Omega_{R/\mathbb{C}},\, R)$. This has been conjectured to be true and is known as the Zariski-Lipman conjecture.

It is known that if the module of derivations is locally free then R must be normal (see [J. Lipman, *Free derivation modules on algebraic varieties*, Amer. J. Math. **87** (1965) 874–898]) and it is known that the conjecture is correct if R is a hypersurface (a regular domain modulo one nonzero element) (see [G. Scheja and U. Storch, *Über differentielle Abhängigkeit bei idealen analytischer Algebren*, Math. Z. **114** (1970) 101–112] and [____, *Differentielle Eigenschaften des Lokaliserungen analytischer Algebren*, Math. Ann. **197** (1972) 137–170]), or if R is graded [M. Hochster, *The Zariski-Lipman conjecture in the graded case*, J. of Alg. **47** (1977) 411–424]. By a result of Flenner [H. Flenner, *Extendability of differential forms on nonisolated singularities*, Invent. Math. **94** (1988) 317–326], if the theorem is true in dimension two then it is true. It remains open in dimension two, even if the ring is a complete intersection in a polynomial ring in four variables.

We return to the problem of defining the intersection multiplicity of a variety X and a linear space Y in $\mathbb{A}^n_{\mathbb{C}}$. Let $\dim(X) = d$. We assume that the intersection is proper, so that $\dim(Y) = n - d$, and we make a change of coordinates so that the point of intersection that we are studying is the origin, which we assume is an isolated point of intersection. Then Y can be defined by the vanishing of d homogeneous linear equations L_j, and we write $Y = V(L_1,\, \ldots,\, L_d)$. For each d-tuple of complex numbers $\underline{\epsilon} = (\epsilon_1,\, \ldots,\, \epsilon_d)$ we let $Y_{\underline{\epsilon}} = V(L_1 - \epsilon_1,\, \ldots,\, L_d - \epsilon_d)$. By the Euclidean topology on $\mathbb{A}^N_{\mathbb{C}}$ we mean the usual metric space topology, which is Hausdorff and locally compact. There is a Euclidean topology on any closed algebraic set, inherited from a copy if $\mathbb{A}^N_{\mathbb{C}}$ in which it is embedded. The topology is independent of the embedding, since isomorphisms of closed algebraic sets are given, in both directions, by polynomial maps in the coordinates, and these will be continuous in the respective Euclidean topologies and therefore provide a homeomorphism.

We next switch points of view. The functions $L_1,\, \ldots,\, L_d$, restricted to X, give d elements of the coordinate ring $\mathbb{C}[X]$. Let $(R, P, \mathbb{C})$ be the local ring of $\mathbb{C}[X]$ at the point we are interested in. The assertion that this point is an isolated point of the intersection means precisely that $L_1,\, \ldots,\, L_d$ is a system of parameters in R: the contraction of P to $\mathbb{C}[X]$ is a minimal prime of $(L_1,\, \ldots,\, L_d)\mathbb{C}[X]$, because the point in question is an irreducible component of the intersection. Moreover, $\dim(R) = \dim(\mathbb{C}[X]) = d$. Consider the map $\Gamma : \mathbb{C}[X] \to \mathbb{A}^d_{\mathbb{C}}$ given in coordinates by $(L_1,\, \ldots,\, L_d)$. This is the same as the map of algebraic sets induced by the $\mathbb{C}$-algebra inclusion $\mathbb{C}[L_1,\, \ldots,\, L_d] \subseteq \mathbb{C}[X]$. Then $X \cap Y_{\underline{\epsilon}}$ is precisely the same as $\Gamma^{-1}(\underline{\epsilon})$. The fact that we want can now be phrased as follows:

Theorem. *Let $\Gamma : X \to \mathbb{A}^d_{\mathbb{C}}$ as described and let x be the point of X that we are studying, so that Γ maps x to the origin $\mathbf{0}$ in $\mathbb{A}^d_{\mathbb{C}}$. Then there are Euclidean neighborhoods B' of x*

and B of $\mathbf{0}$ in $\mathbb{A}^d_C$ and a proper Zariski closed subset Z of $\mathbb{A}^d_C$ such that for all $\underline{\epsilon} \in B - Z$, the cardinality of $\Gamma^{-1}(\underline{\epsilon}) \cap B'$ is a constant μ, and all points in $\Gamma^{-1}(\underline{\epsilon}) \cap B'$ approach x as $\underline{\epsilon}$ approaches $\mathbf{0}$. Moreover μ is the torsion-free rank of $\widehat{R}$ as a module over $\mathbb{C}[[L_1, \ldots, L_d]]$.

It will take us a while to prove this result. Once we have this theorem, we can define the intersection multiplicity of X and Y at x as the value of μ. When Y is not necessarily linear we replace X and Y by $X \times Y$ and Δ.

Let us define the distance between two monic polynomials of degree m as the supremum of the absolute values of differences of corresponding coefficients. By the root set $\mathcal{S}(f)$ of a monic polynomial f, we mean a sequence that gives the roots $\theta_1, \ldots, \theta_m$ of the polynomial, each occurring a number of times equal to its multiplicity, but we consider these sequences only up to equivalence: if π is a permutation of $\{1, \ldots, m\}$ then we regard $\theta_1, \ldots, \theta_m$ as the same root set as $\theta_{\pi(1)}, \ldots, \theta_{\pi(m)}$. If $\mathcal{S}(f)$ is represented by the sequence $\theta_1, \ldots, \theta_m$ and $\mathcal{S}(g)$ is represented by $\theta'_1, \ldots, \theta'_m$, then we define the distance between the root sets as

$$\min_{\pi \in S_m} \sup_{1 \le t \le m} \{|\theta_m - \theta'_{\pi(m)}|\}.$$

Thus, two root sets are close if and only if for some choice of orderings of their terms, the corresponding terms are all close.

Lemma. *Let $f \in \mathbb{C}[z]$ be a monic polynomial of degree $m \ge 1$. Then for all $\epsilon > 0$ there exists $\delta > 0$ such that if g is within distance δ of f then $\mathcal{S}(g)$ is within distance ϵ of $\mathcal{S}(f)$: that is, the root set of f is a continuous function of the coefficients of f as f varies in the set of monic polynomials of degree m. Yet another formulation is that if g_t is a sequence of monic polynomials of degree m that converges to f, then the root sets $\mathcal{S}(g_t)$ converge to $\mathcal{S}(f)$.*

Proof. This is obvious if $m = 1$. We assume that $m > 1$ and use induction. It will suffice to prove the final statement. Choose a root θ of f. Replace f by $f(z+\theta)$ and g_t by $g_t(z+\theta)$. We still have that $g_t(z+\theta) \to f(z+\theta)$, and all root sets have been translated by $-\theta$, so that the distance from the root set of $g_t(z+\theta)$ to the root set of $f(z+\theta)$ is the same as the distance from the root set of g_t to the root set of f. Therefore, we may assume without loss of generality that 0 is a root of f, i.e., that its constant term is 0. Let c_t be the constant term of g_t. By hypothesis $c_t \to 0$ and so $|c_t| \to 0$. It follows as well that $|c_t|^{1/m} \to 0$. Since the product of the absolute values of the roots of g_t is $|c_t|$, we know that g_t has at least one root θ_t with $|\theta_t| \le |c_t|^{1/m}$. Then $\theta_t \to 0$, from which it follows $h_t = g_t(z+\theta_t) \to f$. Each $h(t)$ has 0 as a root, and so $h_t = zh^*_t$ and $f = zf^*$. But then $h^*_t \to f^*$, and $\mathcal{S}(h^*_t) \to \mathcal{S}(f^*)$ by the induction hypothesis. It follows that $\mathcal{S}(h_t) \to \mathcal{S}(f)$. Since the distance between the root sets of h_t and g_t is at most $|\theta_t|$ and $|\theta_t| \to 0$, it follows that $\mathcal{S}(g_t) \to \mathcal{S}(f)$ as well. $\square$

We next want to use this result to prove that if we have a map of closed algebraic sets $G : X \to W$ with $x \in X$ mapping to $w \in W$ and the map is étale near x, i.e., the map $\mathbb{C}[W] \to \mathbb{C}[X]$ is étale near the maximal ideal Q_x of $\mathbb{C}[X]$ corresponding to x, then there are Euclidean open neighborhoods U of x and V of w such that G maps U homeomorphically onto V.