Math 711: Lecture of November 3, 2004

Before proceeding further, we note the following: if one uses a different set of d linearly independent linear equations $L_1', \ldots, L_d'$ to define Y, the number μ obtained by counting points in fibers does not change. Thus, μ does not depend on the choice of the L_i, only on Y.

The reason is that the effect of the change is that one winds up studying $A \circ \Gamma : X \to \mathbb{A}^d_{\mathbb{C}}$ instead of Γ, where $A : \mathbb{A}^d_{\mathbb{C}} \to \mathbb{A}^d_{\mathbb{C}}$ is an invertible linear transformation. Since A is a homeomorphism in both the Zariski and Euclidean topologies, this has no effect on the number of points close to x in fibers over points near $\mathbf{0}$ when one restricts the points considered to the intersection of a Euclidean open neighborhood of $\mathbf{0}$ with a non-empty Zariski open set.

We next prove:

Proposition. *Let $G : X \to W$ be a map of closed algebraic sets of $\mathbb{C}$ and $x \in X$ be such that $G(x) = w \in W$. Suppose that G is étale near x, i.e., that the map $\mathbb{C}[W] \to \mathbb{C}[X]$ is étale near the maximal ideal Q of $\mathbb{C}[X]$ corresponding to x. Then there are Euclidean open neighborhoods U of x and V of w such that G maps U homeomorphically onto V.*

Proof. Let $R = \mathbb{C}[W]$ and $S = \mathbb{C}[X]$. Since S is étale near Q, we may replace R and S by localizations such that the map $R \to S$ is standard étale. Therefore, we may assume without loss of generality that $S = R[z]_g/(f)$, where the image of f' is invertible in S. Think of W as a Zariski closed set in $\mathbb{A}^s_{\mathbb{C}}$. Then $X \subseteq A^{s+1}_{\mathbb{C}}$ and

$$X = \{(v, \lambda) \in W \times \mathbb{A}^1_{\mathbb{C}} : f(v, \lambda) = 0, \ \ g(v, \lambda) \neq 0\}.$$

The map G now corresponds to the product projection on the first coordinate, suitably restricted.

We write $f_v(z)$ for the polynomial $f(z)$ with its coefficients (which are in R and therefore functions on W) evaluated at the point v of W. The coefficients of $f_v(z)$ are continuous functions on W. Let x correspond to (w, θ_w), where θ_w is a root of f_w such that $g_w(\theta_w) \neq 0$.

We also know that at (w, θ_w), the derivative of f_w does not vanish, and so the roots of the monic polynomial $f_w(z)$ giving points where g does not vanish are simple: evaluation at the point gives a map $R[x]_g/(f) \to \mathbb{C}$, and since f' is invertible in $R[x]_g/(f)$, it maps to a nonzero element of $\mathbb{C}$. Let c be less than half the distance between any other root of f_w and θ_w. Then there is a Euclidean open neighborhood V of w such that for all $v \in V$, the distance between the root set of f_v and the root set of f_w is $< c/4$. For every $v \in V$ there is a unique root of f_v within $c/4$ of of θ_w: call this root $\Theta(v)$. Each root of f_v is within distance $c/4$ of a root of f_w, and so any root of f_v other than $\Theta(v)$ is at distance at least $2c - c/4 - c/4 > c$ from $\Theta(v)$.

Because the coefficients of f_v vary continuously with v, so does the root set of f_v, and it follows that $\Theta(v)$ is a continuous function of v. By decreasing V, if necessary, we may assume that $g\big(v, \Theta(v)\big) \neq 0$ for $v \in V$. Then the map H defined by $H(v) = (v, \Theta(v))$, and the product projection on the first coordinate (which is G) are mutually inverse continuous functions. Let $U = H(V)$. It suffices to show that U is open in X, and for this it suffices to show that U is open in $G^{-1}(V)$. But the ball of radius $c/4$ around a point $\big(v, \Theta(v)\big)$ of U is contained in U, for if (v', λ) is in that ball, λ is within $c/4$ of $\Theta(v)$ and therefore with $c/4 + c/4$ of $\Theta(w) = \theta_w$. Therefore, $\Theta(v')$ is within $c/4 + c/4 + c/4 < c$ of λ, which forces $\Theta(v') = \lambda$, and we are done, since $\big(v', \Theta(v')\big) \in U$. $\square$

The next step is to prove the following:

Theorem. *Suppose that $\Gamma : X \to \mathbb{A}^d_{\mathbb{C}}$ and that $x \in X$ is isolated in its fiber over $\mathbf{0} \in \mathbb{A}^d_{\mathbb{C}}$. Then there is a commutative diagram of closed algebraic sets:*

$$\begin{array}{ccc} \widetilde{X} & \xrightarrow{\;G\;} & \widetilde{\mathbb{A}}^d_{\mathbb{C}} \\ \big\downarrow & & \big\downarrow \\ X & \xrightarrow[\;\Gamma\;]{} & \mathbb{A}^d_{\mathbb{C}} \end{array}$$

such that $\widetilde{\mathbb{A}}^d_{\mathbb{C}}$ is smooth and irreducible, the vertical arrows are étale, and G is finite. Moreover, there are points $\widetilde{x} \in \widetilde{X}$ and $\widetilde{\mathbf{0}} \in \widetilde{\mathbb{A}}^d_{\mathbb{C}}$ such that $G(\widetilde{x}) = \widetilde{\mathbf{0}}$, $\widetilde{x} \mapsto x$, $\widetilde{\mathbf{0}} \mapsto \mathbf{0}$, and $\widetilde{x}$ is the only point of $\widetilde{X}$ that maps to $\widetilde{\mathbf{0}}$.

The statement that G is finite means that the corresponding map of rings is module-finite. For the purpose of doing counting of points in fibers we will be able to replace the map Γ by the map G and the points x and $\mathbf{0}$ by $\widetilde{x}$ and $\widetilde{\mathbf{0}}$. A key point is that near these points, each point has a Euclidean neighborhood carried homeomorphically by the appropriate map to a Euclidean open neighborhood of $\mathbf{0}$ or $\widetilde{\mathbf{0}}$. Instead of passing to Euclidean neighborhoods, we have passed to "étale neighborhoods."

We postpone the proof for a bit. We first prove a purely ring-theoretic result that will be helpful.

Theorem. *Let A be a Noetherian ring, R a finitely generated A-algebra, let m be a maximal ideal of A and let $n \subseteq R$ be a maximal ideal of R that is a minimal prime of mR. Then R^{h}_n is module-finite over A^{h}_m, and is a localization of $R \otimes_A A^{\mathrm{h}}_m$ at one element.*

Note that if A and R are finitely generated algebras over $\mathbb{C}$ and m, n are maximal ideals, the residue class fields are both $\mathbb{C}$.

Note that $\widehat{R_n}$ is module-finite over $\widehat{A_m}$: it is easy to see that the extension of residue class fields is finite algebraic, and mR_n is n-primary. This illustrates a frequently recurring phenomenon: a result for completions often has an analogue for Henselizations.

The Theorem above is quite non-trivial: the proof uses Zariski's Main Theorem.