Math 711: Lecture of November 5, 2004

Proof of the Theorem. Note that since R is finitely generated as an A-algebra, R/n is a field finitely generated as an algebra over the field A/m. By one form of Hilbert's Nullstellensatz, this implies that that $L = R/n$ is a finite algebraic extension of $K = A/m$.

Let T be the integral closure of $C = A_m^{\mathrm{h}}$ in $C \otimes_A R$. We have an obvious map

$$C \otimes_A R \twoheadrightarrow C/mC \otimes_R R/n \cong K \otimes_K L \cong L.$$

The kernel of the composite map is evidently a maximal ideal Q of $C \otimes_A R$ lying over the maximal ideal mC of C (mC is the maximal ideal of C because C is the Henselization of (A, mA_m)). The ideal Q is evidently maximal in its fiber over $mC \in \mathrm{Spec}\,(C)$, but it is minimal as well, since

$$C \otimes_A R/(mC)^{\mathrm{e}} \cong C/mC \otimes_A R/mR \cong K \otimes_A R/mR \cong R/mR,$$

and n is minimal over mR. It follows that there exists $t \in T - Q$ such that $T_t = (C \otimes_A R)_t$. Since $C \otimes_A R$ is finitely generated over C by a finite set of generators of R, we can choose a power t^N of t and a subalgebra T_0 of T containing t and module-finite over C such that t^N multiplies the image of every generator of R into T_0. It follows that $(T_0)_t \cong (C \otimes_A R)_t$.

Now, $(C \otimes_A R)_Q$ is a local ring of $(C \otimes_A R)_t$ at a maximal ideal, and so it is also a local ring of $(T_0)_t$ at a maximal ideal. But T_0 is module-finite over the Henselian local ring C, and therefore it decomposes as a product of local rings: one of these must be the same as $(C \otimes_A R)_Q$, Each of these local rings is module-finite over the Henselian ring C via a local map, and, therefore, Henselian: we leave this an exercise. It follows that $(R \otimes_A C)_Q$ is Henselian. It is the localization of $(T_0)_t$ at an idempotent, and it is therefore the localization of T_0 at one element. But then it is also the localization of $C \otimes_A R$ at one element.

We may now complete the proof by showing that $(R \otimes_A C)_Q$ is the Henselization of R. First note that since there is a local R-algebra map $R \to (R \otimes_A C)_Q$ and this ring is Henselian, we have a unique local R-algebra map $R_n^{\mathrm{h}} \to (R \otimes_A C)_Q$. On the other hand, C is a direct limit of pointed étale extensions C_i of A. Each of these, when tensored with R, is a localization of an étale extension of R. Let Q_i be the kernel of the composite map

$$C_i \otimes_A R \twoheadrightarrow K \otimes_K R/n \cong L.$$

Then $(C \otimes_A R)_Q$ is the direct limit of the $(C_i \otimes_A R)_{Q_i}$, each of which is pointed étale over R. But this gives a unique local R-algebra map $(C \otimes_A R)_Q \to R_n^{\mathrm{h}}$, and it is injective. The composition

$$R_n^{\mathrm{h}} \to (C \otimes_A R)_Q \to R_n^{\mathrm{h}}$$

is a local R-algebra map $R_n^{\mathrm{h}} \to R_n^{\mathrm{h}}$, and so is evidently the identity, which forces the injection $(C \otimes_A R)_Q \to R_n^{\mathrm{h}}$ to be surjective as well. $\square$

The next result is aimed at replacing the Henselizations A^{h}_m and R^{h}_n by étale extensions. We want to descend from the Henselizations to pointed étale extensions and then "unlocalize" these, in a manner of speaking. For this result we impose some additional hypotheses, namely that C be a domain and that $\dim(A) = \dim(R)$. C is a domain if A is regular, and these hypotheses will hold in the geometric situation in which we want to apply the result.

Theorem. *Let A, m, R, n be as in the preceding theorem. Let $C = A^{\mathrm{h}}_m$ and $D = R^{\mathrm{h}}_n$. Assume that C is a domain, which is true when A_m is regular, and that $\dim(A_m) = \dim(R_n)$. Then $C \to D$ is injective, and since D is module-finite, it has a torsion-free rank $\mu > 0$ over C. Moreover, $\widehat{D} \cong \widehat{R_n}$ has torsion-free rank μ over $\widehat{C} \cong \widehat{A_m}$.*

Let $\eta \in C \otimes_A R$ be such that $(C \otimes_A R)_\eta \cong D$. Then there are étale extensions $\widetilde{A}$ of A and $\widetilde{R}$ of R such that there is a commutative diagram

$$\begin{array}{ccc} C & \longrightarrow & D \\ \uparrow & & \uparrow \\ \widetilde{A} & \longrightarrow & \widetilde{R} \\ \uparrow & & \uparrow \\ A & \longrightarrow & R \end{array}$$

and such that:

1. *There is a unique maximal ideal $\widetilde{m}$ of $\widetilde{A}$ lying over m in A; moreover, $\widetilde{m} = m\widetilde{A}$ and is the contraction of mC.*
2. *There is a unique maximal ideal $\widetilde{n}$ of $\widetilde{R}$ lying over $\widetilde{m}$ in $\widetilde{A}$ (equivalently, lying over m in A). This maximal ideal is also the unique maximal ideal of $\widetilde{R}$ lying over n in R, and it is the contraction of nD.*
3. *There is an element $\eta_0 \in \widetilde{A} \otimes_A R$ such that η_0 maps to η in $C \otimes_A R$ and $\widetilde{R} = (\widetilde{A} \otimes_A R)_{\eta_0}$.*
4. *$\widetilde{R}$ is a module-finite extension of $\widetilde{A}$ of torsion-free rank μ.*

Proof. We know that $\dim(C) = \dim(A_m)$ and $\dim(D) = \dim(R_n) = \dim(A_m)$, so that $\dim(C) = \dim(D)$. Since D is module-finite over C, it has the same dimension as the image of C. Since C is a domain, $C \to D$ cannot have a kernel, or the dimension of D will be strictly smaller than the dimension of C. Thus, D has some torsion-free rank $\mu > 0$ over C, and there will be an exact sequence

$$0 \to C^\mu \to D \to W \to 0$$

where W is killed by a nonzero element of C. Since completion is exact,

$$0 \to \widehat{C}^\mu \to \widehat{D} \to \widehat{W} \to 0$$

is exact, and it follows that the torsion-free rank of $\widehat{D}$ over $\widehat{C}$ is also μ.

Next observe that C is a directed union of pointed étale extensions of A_m. Here, the maps are faithfully flat and, hence, injective. Each of these pointed étale extensions is a direct limit of localizations of an étale extension at one element, and these are simply étale extensions themselves. Thus, C is a direct limit of étale extensions of A. Each may be localized further at an element to kill the kernel of the map to C if there is any. Thus, we may view C as a directed union of étale extensions $\widetilde{A}$ of A. Each of these may be viewed as a localization at one element of a standard étale extension of A, and so will be the localization at one element of a module-finite extension of A. In a pointed étale extension of A_m, m generates the maximal ideal, which is contracted from mC. It follows that in a suitable localization of $\widetilde{A}$, the expansion of m will be the contraction of mC. We henceforth use $\widetilde{A}$ for an étale extension of A satisfying these conditions: we are free to enlarge the choice of $\widetilde{A}$ further to satisfy some additional conditions.

Choose a finite set of elements v_j of $C \otimes_A R$ whose images generate $(C \otimes_A R)_\eta$ as a C-module. We may further assume, enlarging the set of generators if necessary, that the images of $v_1, \dots, v_\mu$ are linearly independent over C and, and that we have a nonzero element $c \in C$ that kills $(C \otimes_A R)_\eta$ modulo the C-span of the elements $v_1, \dots, v_\mu$.

Now choose $\widetilde{A}$ in the direct limit system of étale extensions of A so large that $\widetilde{A}$ contains an element $\widetilde{c}$ that maps to c, so large that $\widetilde{A} \otimes_A R$ contains an element η_0 that maps to η, and so large that $(\widetilde{A} \otimes_A R)_{\eta_0}$ contains elements $\widetilde{v}_j$ that map to the v_j. Then

$$(\widetilde{A} \otimes_A R)_{\eta_0} / \sum_j \widetilde{A}\widetilde{v}_j$$

is killed when we apply $C \otimes_{\widetilde{A}} \_$, and it follows that if we replace $\widetilde{A}$ by a suitably larger étale extension this quotient will be zero. Similarly, after replacing $\widetilde{A}$ by a suitably larger étale extension, $\widetilde{c}$ will multiply $(\widetilde{A} \otimes_A R)_{\eta_0}$ into $\sum_j \widetilde{A}\widetilde{v}_j$.

Localization of $\widetilde{A}$ at one element will again guarantee that $mC \cap \widetilde{A} = m\widetilde{A}$ is the only maximal ideal of $\widetilde{A}$ lying over m. Consider the corresponding choice of $\widetilde{R} = (\widetilde{A} \otimes_A R)_{\eta_0}$. The contraction of nD to $\widetilde{R}$ lies over m in A, since nD lies over m in A, and therefore lies over $m\widetilde{A}$ in $\widetilde{A}$. The maximal ideals of $\widetilde{R}$ lying over m in A correspond to the maximal ideals of $\widetilde{R}/m\widetilde{R} \cong \big((\widetilde{A}/m\widetilde{A}) \otimes_A R\big)_{\eta_0} \cong (K \otimes_A R)_{\eta_0} \cong (R/mR)_{\eta_0}$. On the other hand, D has a unique maximal ideal, nD, lying over mA^{h}, and we conclude that $\big((A^{\mathrm{h}}/mA^h) \otimes_A R\big)_\eta$ has only one maximal ideal, and this ring becomes $(K \otimes_A R)_\eta \cong (R/mR)_\eta$. Since η_0 maps to η, they have the same image in R/mR. Therefore, $(R/mR)_{\eta_0} = (R/mR)_\eta$, and there is a unique maximal ideal of $\widetilde{R}$ lying over $m\widetilde{A}$. It is the only maximal ideal lying over n, since n lies over m. $\square$