Math 711: Lecture of November 8, 2004

The preceding Theorem now immediately implies the result stated in the Lecture Notes of November 3:

Theorem. *Suppose that* $\Gamma : X \to \mathbb{A}^d_{\mathbb{C}}$ *and that* $x \in X$ *is isolated in its fiber over* $\mathbf{0} \in \mathbb{A}^d_{\mathbb{C}}$. *Then there is a commutative diagram of closed algebraic sets:*

$$\begin{tikzcd} \widetilde{X} \arrow[r, "G"] \arrow[d] & \widetilde{\mathbb{A}}^d_{\mathbb{C}} \arrow[d] \\ X \arrow[r, "\Gamma"'] & \mathbb{A}^d_{\mathbb{C}} \end{tikzcd}$$

such that $\widetilde{\mathbb{A}}^d_{\mathbb{C}}$ *is smooth and irreducible, the vertical arrows are étale, and* G *is finite. Moreover, there are points* $\widetilde{x} \in \widetilde{X}$ *and* $\widetilde{\mathbf{0}} \in \widetilde{\mathbb{A}}^d_{\mathbb{C}}$ *such that* $G(\widetilde{x}) = \widetilde{\mathbf{0}}$, $\widetilde{x} \mapsto x$, $\widetilde{\mathbf{0}} \mapsto \mathbf{0}$, *and* $\widetilde{x}$ *is the only point of* $\widetilde{X}$ *that maps to* $\widetilde{\mathbf{0}}$.

We can now use this to justify our earlier statements about counting of points in fibers. But we first need one more fact:

Proposition. *Let* $g : Y \to Z$ *be a finite morphism of affine algebraic sets over* $\mathbb{C}$, *and suppose that* $y \in Y$ *is the only point lying over* $z \in Z$. *Then there for every Euclidean neighborhood* U *of* y *in* Y *there is a Euclidean neighborhood* V *of* z *in* Z *such that for all points* $v \in V$, $g^{-1}(v) \subseteq U$.

Proof. If not, we can choose a real number $\epsilon > 0$, a sequence of points $\{z_i\}_i$ in Z converging to z, and points $\{y_i\}_i \in Y$ such that for all i, $g(y_i) = z_i$ while $|y_i - y| > \epsilon$. Think of Y as embedded in $\mathbb{A}^N_{\mathbb{C}}$ and let $F_1, \ldots, F_N$ be coordinate functions on Y. Since $\mathbb{C}[Y]$ is module-finite over $\mathbb{C}[Z]$, each F_j satisfies a monic polynomial equation over $\mathbb{C}[Z]$, say

$$F_j^{d_j} + \sum_{t=0}^{d_j-1} r_{j,t} F_j^t = 0,$$

where the $r_{j,t} \in \mathbb{C}[Z]$.

This implies that the coordinates of the y_i are bounded. To see this for $F_j(y_i)$, note that it is in the root set of $F_j^{d_j} + \sum_{t=0}^{d_j-1} r_{j,t}(z_i) F_j^t = 0$: as $z_i \to z$, this root set converges to the root set of $F_j^{d_j} + \sum_{t=0}^{d_j-1} r_{j,t}(z) F_j^t = 0$. It follows that the sequence $\{y_i\}_i$ has a subsequence that converges, in the Euclidean topology, to a point $y' \in \mathbb{A}^N_{\mathbb{C}}$. Since Y is closed in the Zariski topology in $\mathbb{A}^N_{\mathbb{C}}$, it is closed in the Euclidean topology, and $y' \in Y$. Change notation, replacing the original sequence of y_i by this subsequence and taking the corresponding subsequence of the z_i. Then $y' \neq y$, but $g(y') = z$, by the continuity of g in the Euclidean topology, a contradiction. $\square$

We are now ready to justify the statement we made earlier about counting points in fibers. The torsion-free rank μ of $\mathbb{C}[\widetilde{X}]$ over $\mathbb{C}[\widetilde{\mathbb{A}}^d]$ is the same as the degree of the extension of function fields. By part (c) of the Lemma on the third page of the Lecture Notes of September 8, off a proper Zariski closed subset Z' of $\widetilde{\mathbb{A}}^d$, all fibers have μ distinct points. Since the map is finite, by the preceding Proposition all points of any fiber over a point close to $\widetilde{\mathbf{0}}$ are approaching $\widetilde{x}$, and off Z' there are always μ such points. Thus, for any small Euclidean neighborhood U of $\widetilde{x}$ there is a small Euclidean neighborhood V of $\widetilde{\mathbf{0}}$ such that all fibers over points of $V - Z'$ are contained in U and have cardinality μ. But, since the vertical arrows are étale, by the first Proposition of the Lecture Notes of November 3, sufficiently small neighborhoods of $\widetilde{x}$ and $\widetilde{\mathbf{0}}$ respectively will be carried homeomorphically to Euclidean open neighborhoods of x and $\mathbf{0}$, respectively. The image of Z' will be constructible of dimension smaller than d, and so its closure Z will be a proper closed subset of $\mathbb{A}^d_C$. This establishes our earlier statements about counting points in fibers. Note that μ will also be the torsion-free rank of the Henselization of the local ring at x over the Henselization of the local ring at $\mathbf{0}$ (or we may use $\widetilde{x}$ and $\widetilde{\mathbf{0}}$: the Henselizations don't change), and will also be the torsion-free rank of the completion of the local rings at x over the completion of the local ring at $\mathbf{0}$: the latter is a regular local ring.

We next want to relate multiplicities thought of in this geometric way with Serre's definition using alternating sums of lengths of Tor.

Let A be the completion of the local ring at $\mathbf{0}$: $L_1, \ldots, L_d$ is a regular system of parameters. Let $\underline{L} = L_1, \ldots, L_d$. Let R be the completion of the local ring at x. Let M be any finitely generated A-module. We first observe that the torsion-free rank of M over A may also be obtained as

$$\chi(\underline{L}; M) = \sum_{i=0}^{d} (-1)^i \ell\bigl(H_i(\underline{L}; M)\bigr)$$

where $H_i(\underline{L}; M)$ indicates Koszul homology. To check that the rank of M is the same as $\chi(\underline{L};\, M)$, first note that both are additive on short exact sequences of finitely generated modules. From this it follows that if

$$0 \to A^{b_d} \to \cdots \to A^{b_1} \to A^{b_0} \to M \to 0$$

is a free resolution of M, then the rank of M is $b_0 - b_1 + \cdots + (-1)^d b_d$ times the rank of A, which is 1, and it is also $b_0 - b_1 + \cdots + (-1)^d b_d$ times $\chi(\underline{L}; A)$. But this is also 1: because $L_1, \ldots, L_d$ is a regular sequence, the Koszul complex is a free resolution of $A/(\underline{L}) \cong \mathbb{C}$, the residue class field. All higher Koszul homology is 0, and $H_0(\underline{L}; A) \cong \mathbb{C}$, from which $\chi(\underline{L}; A) = 1$ follows. Note also that the Koszul homology modules are the same as the modules $\mathrm{Tor}_i^A(\mathbb{C}, M)$, since the Koszul complex resolves $\mathbb{C} \cong A/(\underline{L})$.

Now suppose that we are considering varieties $X = V(P)$ and $Y = V(Q)$ with an isolated point of intersection x. For convenience we translate coordinates so that x is the origin. Our geometric method of getting at the intersection multiplicity is to work with $X \times Y$ and Δ, the diagonal, instead. The completed local ring of $X \times Y$ at (x, x) is the complete tensor product over $\mathbb{C}$ of the completed local rings A/PA and A/QA of X and Y, respectively. Let $f_1, \ldots, f_d$ be the images of the defining linear forms of the diagonal in $(A/PA)\widehat{\otimes}_{\mathbb{C}}(A/QA)$. From the remarks above, μ is

$$\sum_{i=0}^{d}(-1)^i\ell\big(H_i(f_1, \ldots, f_d; (A/PA)\widehat{\otimes}_{\mathbb{C}}(A/QA))\big).$$

Since $f_1, \ldots, f_d$ is a regular sequence in $B = A\widehat{\otimes}_{\mathbb{C}}A$, this is

$$\sum_{i=0}^{d}(-1)^i\ell\big(\mathrm{Tor}_i^B(A, (A/PA)\widehat{\otimes}_{\mathbb{C}}(A/QA))\big),$$

where the copy of A on the left in Tor is $A\widehat{\otimes}_{\mathbb{C}}A/(f_1, \ldots, f_d) \cong A$.

We can calculate this as follows. Take finite free resolutions of A/PA and A/QA over A, and form a double complex by taking their complete tensor product over $\mathbb{C}$. The total complex of this double complex is a free resolution of $(A/PA)\widehat{\otimes}_{\mathbb{C}}(A/QA)$ over $B = A\widehat{\otimes}_{\mathbb{C}}A$ because complete tensor product over $\mathbb{C}$ is exact. When we apply $A\otimes_B \_$ we get a complex whose homology is $\mathrm{Tor}_i^B(A, (A/PA)\widehat{\otimes}_{\mathbb{C}}(A/QA))$. But this complex is also obtained by taking the total complex of the ordinary tensor product over A of the resolutions of A/PA and A/QA over A. Thus,

$$\mathrm{Tor}_i^B(A, (A/PA)\widehat{\otimes}_{\mathbb{C}}(A/QA)) \cong \mathrm{Tor}_A^i(A/PA, A/QA).$$

But then

$$\mu = \sum_{i=0}^{d}(-1)^i\ell\big(\mathrm{Tor}_A^i(A/PA, A/QA)\big).$$

and there is no need to use reduction to the diagonal in the definition.

One more wrinkle: because the point of intersection is isolated, $P+Q$ is primary to the maximal ideal that corresponds to that point. This means not only that the Tor modules in this formula for μ have finite length, but also that the formula for μ is valid if we use the local ring at x instead of the completed local ring A at x throughout the formula: these finite length modules don't change when we complete.