Math 711: Lecture of November 10, 2004

We return to the study of approximation rings.

Proposition. *If R is an approximation ring and I is a proper ideal of R, then R/I is an approximation ring.*

Proof. Let $I = (r_1, \ldots, r_s)R$. Given a system of polynomial equations

$$F_j(X_1, \ldots, X_n) = 0, \quad 1 \leq j \leq m$$

with coefficients in R/I, we may lift coefficients to obtain corresponding polynomials $\widetilde{F}_j(X_1, \ldots, X_n)$ over R. Introduce additional variables Y_{ik} and consider the system

$$(*) \quad \widetilde{F}_j(X_1, \ldots, X_n) - \sum_{i=1}^{s} r_i Y_{ij} = 0$$

$1 \leq j \leq m$ over R. If the F_j have a solution in the completion of R/I, which may be identified with $\widehat{R}/I\widehat{R}$, then the system $(*)$ has a solution in $\widehat{R}$. Thus, $(*)$ has a solution in R, and its image in R/I gives the required solution of the $F_j(X_1, \ldots, X_n) = 0$ in R/I. $\square$

Proposition. *If $R \to S$ is local, where R is an approximation ring and S is module-finite over R, then S is an approximation ring.*

Proof. The idea of the proof is to "push down" a given system of equations over S to a system over R (in different variables) such that solving the original system over S (respectively, $\widehat{S}$) is equivalent to solving the new system over R (respectively, $\widehat{R}$).

Let $\theta_1, \ldots, \theta_h$ be elements of S that span S over R. Let $\bigl(r_{ij}\bigr)$ be an $N \times h$ matrix whose rows span the module of relations over R on the elements $\theta_1, \ldots, \theta_h$. For all i, j there are elements $b_{ijk} \in R$ such that

$$(\#) \quad \theta_i \theta_j = \sum_{k=1}^{h} b_{ijk} \theta_k.$$

Notice that $\theta_1, \ldots, \theta_h$ also span $\widehat{S}$ over $\widehat{R}$, that the rows of the matrix $\bigl(r_{ij}\bigr)$ still span the relations over $\widehat{R}$ (by the right exactness of tensor), and that the equations $(\#)$ hold for $\widehat{R}$ and $\widehat{S}$.

Any element of S (respectively, $\widehat{S}$) can be written as an R-linear (respectively, as an $\widehat{R}$-linear) combination of the elements $\theta_1, \ldots, \theta_h$. Instead of seeking the elements of S that satisfy the equations F_j directly, we look for the coefficients needed to write them as linear combinations of $\theta_1, \ldots, \theta_h$.

We therefore introduce nh new variables Y_{ki} (these will take values in R or $\widehat{R}$). We substitute $\sum_{k=1}^{h} Y_{ki}\theta_k$ for X_i in each equation $F_j = 0$. Using the equations $(\#)$ repeatedly, we can rewrite

$$F_j\Bigl(\sum_{k=1}^{h} Y_{k1}\theta_k, \ldots, \sum_{k=1}^{h} Y_{kn}\theta_k\Bigr)$$

in the form

$$G_{j1}\theta_1 + \cdots + G_{jh}\theta_h$$

where every G_{jt} is a polynomial in the variables Y_{ki} with coefficients in R. The condition that this vanish after we substitute elements of R (or $\widehat{R}$) for the Y_{ki} is that the value of

$$(G_{j1}, \dots, G_{jh})$$

be in the span of the rows of $\bigl(r_{ij}\bigr)$. Let $\rho_1, \dots, \rho_N$ denote the rows of this matrix, and let Z_{jt} be mN new indeterminates. Solving the original system over S (or $\widehat{S}$) is equivalent to solving the system

$$(G_{j1}, \dots, G_{jh}) - \sum_{t=1}^{N} Z_{jt}\rho_t = 0$$

over R (or $\widehat{R}$). This becomes a system of polynomial equations over R if we equate the entries of the vectors on the left to 0 for every j.

A solution of the original system $\widehat{S}$ gives a solution of the new system of $\widehat{R}$. Since R is an approximation ring we get a solution of the new system over R, and this yields the required solution of the original system over S. $\square$

Our next objective is to prove the following:

Lemma. *Let R denote the localization at a prime ideal of a finitely generated algebra over a field K or a DVR (V, tV). Then R^{h} is a homomorphic image of a module-finite local extension of A^{h}, where A is a regular local ring that has the form $\Lambda[\underline{x}]_Q$, where Λ is either a field or else a DVR that is a localization of a finitely generated V-algebra, $\underline{x} = x_1, \dots, x_n$, and Q is generated by the maximal ideal of Λ and $\underline{x}$.*