Math 711: Lecture of November 12, 2004

Before proving the result stated at the end of the Lecture Notes from November 10, we need the following:

Proposition. *Let T be a module-finite extension of a quasilocal ring (R, P, K) and let Q be a maximal ideal of T. Then T^{h}_Q is a local ring of $R^{\mathrm{h}} \otimes_R T$, and the map $R^{\mathrm{h}} \to T^{\mathrm{h}}$ is module-finite and local.*

Proof. Because R^{h} is Henselian and $R^{\mathrm{h}} \otimes_R T$ is module-finite over R_h, it decomposes. The maximal ideals of this ring all contract to the maximal ideal P of R, and so correspond bijectively to the maximal ideals of

$$(R/P) \otimes_R (R^{\mathrm{h}} \otimes_R T) \cong K \otimes_R T \cong T/PT,$$

and, hence, to the maximal ideals of T. Suppose that $\mathcal{Q}$ is the maximal ideal of $R^{\mathrm{h}} \otimes_R T$ corresponding to Q. Then

$$T_Q \to (R^{\mathrm{h}} \otimes_R T)_{\mathcal{Q}}$$

is a local map. Since R^{h} is Henselian, $R^{\mathrm{h}} \otimes_R T$ decomposes, and $(R^{\mathrm{h}} \otimes_R T)_{\mathcal{Q}}$ is one of the factors. Since direct product and direct sum of finitely many modules are the same, $(R^{\mathrm{h}} \otimes_R T)_{\mathcal{Q}}$ is a direct summand of a module-finite extension of R^{h}, and is local. By the first problem of Problem Set #3, it follows that $(R^{\mathrm{h}} \otimes_R T)_{\mathcal{Q}}$ is Henselian. We therefore have a local map

$$T^{\mathrm{h}}_Q \to (R^{\mathrm{h}} \otimes_R T)_{\mathcal{Q}}.$$

Since R^{h} is a direct limit if pointed étale extensions of R, we have that $(R^{\mathrm{h}} \otimes_R T)_{\mathcal{Q}}$ is a direct limit of pointed étale extensions of T_Q, which provides an injective local map $(R^{\mathrm{h}} \otimes_R T)_{\mathcal{Q}} \to T^{\mathrm{h}}_Q$. Since the composite

$$T^{\mathrm{h}}_Q \to (R^{\mathrm{h}} \otimes_R T)_{\mathcal{Q}} \to T^{\mathrm{h}}_Q$$

is a local map that is the identity on T_Q, it is the identity, and so $(R^{\mathrm{h}} \otimes_R T)_{\mathcal{Q}} \to T^{\mathrm{h}}_Q$ is surjective as well as injective. $\square$

We next prove the result stated at the end of the Lecture Notes from November 10, which state again here.

Lemma. *Let R denote the localization at a prime ideal of a finitely generated algebra over a field K or a DVR (V, tV). Then R^{h} is a homomorphic image of a module-finite local extension of A^{h}, where A is a regular local ring that has the form $\Lambda[\underline{x}]_Q$, where Λ is either a field or else a DVR that is a localization of a finitely generated V-algebra, $\underline{x} = x_1, \ldots, x_n$, and Q is generated by the maximal ideal of Λ and $\underline{x}$.*

Proof. Write $R = T_P$ where T is a finitely generated K or V-algebra and P is prime. Since T is a homomorphic image of a domain (even a polynomial ring) finitely generated over K or V, we may assume that T is a domain. In the case of a DVR V, we might as well assume that P contains t: otherwise, we can replace T and V by T_t and V_t, a field, and we are in the case where the base is a field.

The residue class field of R may be assumed to be a finitely generated field over K (in the DVR case, $K = V/tV$). Choose $z_1, \dots, z_h$ in T such that the images of these elements form a transcendence basis for R/PR over K. If $K \subseteq T$ then $K[z_1, \dots, z_h] \subseteq T$ and the elements $z_1, \dots, z_h$ are algebraically independent over K since this is true even mod P. Since the nonzero elements are not in P, they all have inverses, and so

$$\Lambda = K(z_1, \dots, z_h) \subseteq R.$$

Similarly, if T is finitely generated over V then $z_1, \dots, z_h$ are algebraically independent over V, and every element of $V[z_1, \dots, z_h]$ not in (t) has an inverse in R. Let $V(z_1, \dots, z_h)$ denote the localization of $V[z_1, \dots, z_h]$ at the prime ideal (t). Then

$$\Lambda = V(z_1, \dots, z_h) \subseteq R,$$

and Λ is a DVR with maximal ideal generated by (t). We may then replace K or V by Λ, and think of T as a finitely generated Λ-algebra.

In this way we may assume without loss of generality that R/P is a finite algebraic extension of K (which is $\Lambda/t\Lambda$ in the DVR case), so that P is a maximal ideal. We also know that when Λ is a DVR, P contains t. We map a polynomial ring

$$T = \Lambda[Y_1, \dots, Y_n]$$

onto R. Let $\widetilde{P}$ be the maximal ideal that is the inverse image of P. Modulo $\widetilde{P}$, the image of every Y_j is algebraic over K, which is the image of Λ in the DVR case. For each j, let $F_j(W_j)$ denote a monic polynomial in one variable W_j such that the coefficients, other than the leading coefficient, are in K or Λ, that lifts the polynomial over K satisfied by Y_j. Then $x_j = F_j(Y_j) \in \widetilde{P}$. We now see that T is module-finite over $\Lambda[\underline{x}]$, because for $1 \leq j \leq n$, Y_j satisfies the monic polynomial $F_j(Y_j) - x_j = 0$ (here, $-x_j$ becomes part of the constant term of this polynomial). It follows that the x_j are algebraically independent over Λ. The contraction of $\widetilde{P}$ contains t and all the x_j, and so must be equal to $Q = (t, \underline{x})\Lambda[\underline{x}]$. Since T is module-finite over $\Lambda[\underline{x}]$, $(T_{\widetilde{P}})^{\mathrm{h}}$ is module-finite over $\Lambda[\underline{x}]^{\mathrm{h}}_{Q}$, by the preceding Proposition. $\square$