Math 711: Lecture of November 17, 2004

We shall prove a generalized version of the theorem we stated last time, which is given below, but before doing so we want to make a comment about a strengthening of the usual implicit function theorem that requires no effort to prove.

Let (R, m) be a Henselian quasilocal ring and let $\mathfrak{A} \subseteq m$ be a proper ideal of R. Suppose that one has a system of n equations in n unknowns $X_1, \ldots, X_n$, say $F_j(X) = 0$, $1 \le j \le n$. Suppose that one has a solution of these equations, say $\overline{y}_1, \ldots, \overline{y}_n$ in $R/\mathfrak{A}$, and also suppose that the image of the Jacobian determinant

$$\det\Big(\frac{\partial F_j}{\partial X_i}\Big)$$

evaluated at $(\overline{y}_1, \ldots, \overline{y}_n)$ is an invertible element of $R/\mathfrak{A}$. Then the equations have a solution $y_1, \ldots, y_n$ in R, and for all j, y_j is congruent to $\overline{y}_j$ mod $\mathfrak{A}$. The point here is that the solution mod $\mathfrak{A}$ obviously gives a solution mod m, and so the usual implicit function theorem characterization gives a unique solution $(y_1, \ldots, y_n)$ in R lifting the given solution. But the usual implicit function theorem also tells us that the solution in $R/\mathfrak{A}$ is unique (given what it becomes mod m), and so the images of the y_j in $R/\mathfrak{A}$ must be the $\overline{y}_j$. $\square$

Theorem (Tougeron's implicit function theorem). *Let (R, m) be a Henselian quasilocal ring and let $F_1, \ldots, F_h \in R[Y_1, \ldots, Y_d]$. Let $\mathfrak{A} \subseteq m$ be any ideal. Let $\mathcal{J} = \mathcal{J}(Y)$ denote the the $h \times d$ matrix $\big(\partial F_i/\partial Y_j\big)$, which is the transpose of the Jacobian matrix. Let $y'_1, \ldots, y'_d \in R$. Let $\Delta(y')$ denote an ideal of R that annihilates the cokernel of the linear map $R^d \to R^h$ with matrix $\mathcal{J}(y')$, and suppose that*

$$F_j(y') \equiv 0 \bmod \Delta(y')^2\mathfrak{A}.$$

Then there are elements $y_1, \ldots, y_d \in R$ such that

$$F_j(y) = 0,\ 1 \le j \le h, \textit{ and } y \equiv y' \bmod \Delta(y')\mathfrak{A}.$$

Proof. Let $\delta_1, \ldots, \delta_\nu$ generate $\Delta(y')$. Let e_j be a column of the size h identity matrix. Then

$$\delta_i e_j = \mathcal{J}(y')\rho_j^{(i)}$$

for some $\rho_j^{(i)}$. The $\rho_j^{(i)}$ give the columns of a $d \times h$ matrix M_i such that

$$\mathcal{J}(y')M_i = \delta_i \mathbf{1}.$$

Every $F_k(y')$ has the form $\sum_{i,j} \delta_i\delta_j\alpha_{ijk}$ for suitable $\alpha_{ijk} \in \mathfrak{A}$. We shall write α_{ij} for the vector $(\alpha_{ij1}, \ldots, \alpha_{ijh})$.

We seek values for variables Z_{ij} in $\mathfrak{A}$ such that

$$F_k(y' + \sum_{i=1}^{\nu} \delta_i Z_i) = 0$$

where Z_i is the vector $(Z_{i1}, \ldots, Z_{id})$. We write F for $(F_1, \ldots, F_h)$. By Taylor's formula, the equations become

$$F(y') + \mathcal{J}(y') \sum_i \delta_i Z_i + \sum_{ij} \delta_i \delta_j H_{ij}(Z) = 0$$

where the $H_{ij}(Z)$ are vectors of polynomials in the variables Z all of whose terms are degree two and higher.

Using that $F(y') = \sum_{ij} \delta_i \delta_j \alpha_{ij}$ we can write this as

$$\mathcal{J}(y') \sum_i \delta_i Z_i + \sum_{ij} \delta_i \delta_j \big(H_{ij}(Z) + \alpha_{ij}\big) = 0$$

or, since $\delta_j \mathbf{1} = \mathcal{J}(y') M_j$,

$$\sum_i \delta_i \mathcal{J}(y') Z_i + \sum_i \Big(\delta_i J(y') \sum_j M_j \big(H_{ij}(Z) + \alpha_{ij}\big)\Big) = 0$$

or

$$\sum_i \delta_i \mathcal{J}(y') \Big(Z_i + \sum_j M_j \big(H_{ij}(Z) + \alpha_{ij}\big)\Big) = 0$$

This, it suffices to find Z_i with entries in $\mathfrak{A}$ such that

$$Z_i + \sum_j M_j \big(H_{ij}(Z) + \alpha_{ij}\big) = 0.$$

These are vector equations: there are hd equations in hd unknowns. Mod m, the Jacobian matrix with respect to the Z_{ij}, evaluated at $Z_{ij} = 0$, is the identity, since the H_{ij} have only terms of degree 2 or more in the Z_{ij}. Since $Z_{ij} = 0$ is a solution mod $\mathfrak{A}$, by the remarks preceding the statement of the theorem we get a solution in R such that the z_{ij} are in $\mathfrak{A}$. $\square$

We next want to see that in the context of our continuing proof of the Artin approximation theorem, we may assume that one of the minors of the Jacobian matrix has nonzero image in $\widehat{A}$. To see this, we make use of the theory of separable algebras.