Math 711: Lecture of November 22, 2004

We retain our earlier notations: (V, tV, K) is either a DVR or field (in the latter case, $V = K$ and $t = 0$), $T = V[x_1, \dots, x_n]$, $A = T_{(t,x_1,\dots,x_n)}$, $s_1, \dots, s_d \in \widehat{A}$, $S = T[s_1, \dots, s_d] \subseteq \widehat{A}$, P is the kernel of the map

$$\phi : T[Y] = T[Y_1, \dots, Y_d] \twoheadrightarrow S \subseteq \widehat{A}$$

such that $Y_i \mapsto s_i$ for all i, $1 \le i \le d$, $F_1, \dots, F_h \in T[Y]$ are minimal generators of PT_P, $\delta(x, Y)$ is an $h \times h$ minor of the Jacobian matrix $\left(\dfrac{\partial F_j}{\partial Y_i}\right)$ whose image in $\widehat{A}$ is not in $t\widehat{A}$. Let $R = A^{\mathrm{h}}$ and $\mathfrak{A} = m^N$ in the Tougeron Implicit Function Theorem. Then the Theorem shows that in order to complete the proof of the Artin Approximation Theorem, it suffices to prove that for every positive integer N there exists $y' = y'_1, \dots, y'_d \in A^{\mathrm{h}}$ such that

$$(*) \qquad y' \equiv s \bmod m^N \widehat{A}$$

and

$$(**) \qquad F_j(y') \equiv 0 \bmod \delta^2(x, y')m^N.$$

The reason is that the Tougeron Implicit Function Theorem enables us to pass from the solution of the equations $F_j \equiv 0 \bmod \delta^2(x, y')m^N$ to an "honest" solution of the equations $F_j = 0$. The next idea is this: we prefer to worry about a solution mod $\delta^2(x, y')$ and not to worry about the factor m^N, which can be handled automatically by a trick.

The following Lemma permits this reduction:

Key Lemma. *Suppose that $V[x_1, \dots, x_\nu]^{\mathrm{h}}_{(t,x)}$ is an approximation ring for $\nu < n$. Suppose that we have elements $g, F_1, \dots, F_h$ and $s_1, \dots, s_d \in \widehat{A}$ such that t does not divide $g(x, s)$ in $\widehat{A}$ while $g(x, s)$ divides $F_j(x, s)$ in $\widehat{A}$ for $1 \le j \le h$.*

Then for every positive integer N there exists $\underline{y}_N = y_{1N}, \dots, y_{dN} \in (A^{\mathrm{h}})^d$ such that $F_j(x, \underline{y}_N)$ is divisible by $g(x, \underline{y}_N)$ for all j and $s \equiv \underline{y}_N \bmod m^N\widehat{A}$.

We postpone the proof for a while. We first want to see why this Key Lemma enables us to prove the theorem.

Let $\widehat{m} = m\widehat{A}$ and for $u \in \widehat{A} - \{0\}$ let $\mathrm{ord}\,(u)$ denote the largest integer b such that $u \in \widehat{m}^b$, i.e., $u \in \widehat{m}^b - \widehat{m}^{b+1}$. Because the associated graded ring $\mathrm{gr}_{\widehat{m}}\widehat{A}$ is a polynomial ring, and, in particular, a domain, we have that $\mathrm{ord}\,(uv) = \mathrm{ord}\,(u) + \mathrm{ord}\,(v)$ for $u, v \in \widehat{A} - \{0\}$. Note that these remarks apply to any regular local ring, so that we have corresponding notions of order in A and A^{h}. The set of powers of the maximal to which an element of a regular local ring belongs does not change when we pass from the local ring to its completion. Thus, the order of an element is not affected by whether we think of it as being in A, A^{h}, or $\widehat{A}$.

Let $\operatorname{ord}\big(g(x,s)\big) = N_0$. Then for all $N > N_0$, $g(x, \underline{y}_N) \equiv g(x,s) \bmod m^N\widehat{A}$, from which it follows that for all $N > N_0$, $\operatorname{ord}\big(g(x,\underline{y}_N)\big) = N_0$. Since $F_j(x,\, s) = 0$ and $\underline{y}_N \equiv s \bmod m^N\widehat{A}$, we have that for all j, and all N,

$$F_j(x, \underline{y}_N) \in m^N\widehat{A} \cap A^{\mathrm{h}} = m^N A^{\mathrm{h}}.$$

But then for all j and all $N > 0$,

$$F_j(x, \underline{y}_{N+N_0}) \in g(x, \underline{y}_{N+N_0})A^{\mathrm{h}} \cap m^{N+N_0}A^{\mathrm{h}} \subseteq g(x, \underline{y}_N) m^N A^{\mathrm{h}},$$

as required, for an element of order smaller than N cannot multiply an element of order N_0 into $m^{N+N_0}A^{\mathrm{h}}$. $\square$

We shall soon return to the proof of the Key Lemma. In that proof, we shall want to use the Weierstrass Preparation Theorem to keep track of divisibility by g. It will be helpful if g is a regular element in x_n. We can use the following result to arrange this.

Lemma. *Let (V, tV, K) be a DVR or a field, so that we allow the possibility that $t = 0$ and $V = K$. Let n be a positive integer, and let $g \in V[[x_1, \ldots, x_n]]$, the formal power series ring. Suppose that $g \notin (t)$. Then there exist positive integers $b_1, \ldots, b_{n-1}$ such that the continuous V-automorphism mapping $x_i \mapsto x_i + x_n^{b_i}$, $1 \le i \le n-1$, $x_n \mapsto x_n$ takes g to an element that is regular in x_n, i.e., that is not in the ideal $(t,\, x_1, \ldots, x_{n-1})$.*

Proof. In proving this result we may work mod (t) without loss of generality. Therefore, we assume that $t = 0$, i.e., that $V = K$ in the rest of the argument. Our hypothesis on g becomes that g is a nonzero element of $K[[x_1, \ldots, x_n]]$.

We use induction on n. If $n = 1$ it is obvious from the fact that $g \neq 0$ that it is regular in x_1. Note, that quite generally, the condition that we need on $b_1, \ldots, b_{n-1}$ is simply that $g(x_n^{b_1}, \ldots, x_n^{b_{n-1}}, x_n) \neq 0$, for this is what the image of g under the automorphism becomes mod $(x_1, \ldots, x_{n-1})$.

If $n = 2$ we seek b such that $g(x_2^b, x_2) \neq 0$. This is equivalent to the condition that g not be divisible by $x_1 - x_2^b$. The elements $x_1 - x_2^b$ for $b = 1, 2, 3, \ldots$ are prime in the UFD $K[[x_1, x_2]]$, and no two are associates. If g is divisible by all of the elements $x_1 - x_2^b$ for $1 \le b \le B$, then it is divisible by the product of those elements, and is therefore in $(x_1, x_2)^B$. Since $g \neq 0$, it cannot be in $(x_1, x_2)^B$ for all B.

If $n \ge 3$, we may write $g = \sum_{j=i}^{\infty} \gamma_i x_n^j$ where all of the $\gamma_j \in K[[x_1, \ldots, x_{n-1}]]$ and $\gamma_i \neq 0$. By the induction hypothesis we may choose $b_1, \ldots, b_{n-2}$ such that

$$\gamma_i(x_{n-1}^{b_1}, \ldots, x_{n-1}^{b_n-2}, x_{n-1}) \neq 0.$$

Then $g(x_{n-1}^{b_1}, \ldots, x_{n-1}^{b_n-2}, x_{n-1}, x_n)$ is a nonzero formal power series in two variables, and by the case $n = 2$ we can substitute $x_{n-1} = x_n^b$ for some positive integer b and get a nonzero result. $\square$

Notice that an automorphism of $\widehat{V}[[x_1, \ldots, x_n]] = \widehat{A}$ of the form described in the preceding result, as well as its inverse, stabilize A, and hence also induce mutually inverse automorphisms of A^{h}. We henceforth assume that $g = g(x, s)$ is regular in x_n, and we let $\alpha = \alpha(x_n)$ denote its unique monic associate, so that

$$\alpha(x_n) = x_n^q + u_{q-1}x_n^{q-1} + \cdots + u_0$$

where the u_i are nonunits of $C = \widehat{V}[[x_1, \ldots, x_{n-1}]]$.

We want to convert our divisibility problem into an equational problem over

$$D = V[x_1, \ldots, x_{n-1}]^{\mathrm{h}}_{(t, x_1, \ldots, x_{n-1})},$$

which we know from the induction hypothesis is an approximation ring.

Towards this end, we introduce dq new variables $Z_{i,\nu}$ and substitute

$$Z_i = \sum_{\nu=0}^{q-1} Z_{i,\nu} x_n^\nu$$

for the Y_i in the $F_j(x, Y)$ and in $g(x, Y)$ to obtain $F_j(x, Z)$ and $g(x, Z)$. We also introduce new indeterminates $U_0, \ldots, U_{q-1}$ and a variable polynomial H in x_n with these coefficients, so that

$$H(x_n) = H(U, x_n) = x_n^q + \sum_{j=0}^{q-1} U_j x_n^j.$$

Let

$$\mathcal{T} = V[x_1, \ldots, x_{n-1}, Z_{i,\nu}, U_\nu].$$

Working in the polynomial ring $\mathcal{T}[x_n]$ we can formally divide $g(x, Z)$ and $F_j(x, Z)$ by $H(x_n)$ to obtain

$$(*) \qquad g(x, Z) = H(x_n)Q_0 + \sum_{\nu=0}^{q-1} W_{0,\nu} x_n^\nu$$

$$(**) \qquad F_j(x, Z) = H(x_n)Q_j + \sum_{\nu=0}^{q-1} W_{j,\nu} x_n^\nu$$

where the $Q_j \in \mathcal{T}[x_n]$ and the $W_{j,\nu} \in \mathcal{T}$ do *not* involve the variable x_n for all $j \geq 0$.

By the Weierstrass Preparation Theorem we can divide every s_i by $\alpha(x_n)$, the unique monic associate of $g = g(x, s)$, to obtain:

$$(\#) \qquad s_i = \alpha(x_n)\theta_i + \sum_{\nu=0}^{q-1} z_{i,\nu} x_n^\nu,$$

where $\theta_i \in \widehat{A}$ and the $z_{i,\nu} \in C$. We define

$$z_i = \sum_{\nu=0}^{q-1} z_{i,\nu} x_n^\nu.$$