Math 711: Lecture of December 1, 2004

Before proving the Theorem stated in the previous lecture, we introduce some ideas that will simplify the problem. We replace T by its localization at tT, which we call Λ_0. Then $(\Lambda_0, t\Lambda_0, L_0)$ is a DVR, and we have a local map $\Lambda_0 \to \Lambda$: we use L for the residue class field of Λ. We replace S by $\Lambda_0 \otimes_T S$, which is a localization of S and a subring of Λ. Note that this ring is $\Lambda_0[s_1, \ldots, s_d]$ and has the presentation

$$\Lambda_0[Y_1, \ldots, Y_d]/(F_1, \ldots, F_k).$$

The image of Jacobian matrix $\mathcal{J}$ is literally the same, although we now think of the entries as being in $\Lambda_0 \otimes_T S$, the "new" S, which we temporarily denote as S°. Again, we may use $\mathcal{J}'$ for the same matrix over Λ. We write $\mathcal{F}$ and $\mathcal{G}$ for the fraction fields of Λ_0 and Λ respectively.

The theory of excellent rings implies that the field extensions $\mathcal{F} \to \mathcal{G}$ and $L_0 \to L$ are separable. For the new S° we again have an operation of enlargement: if $Q = t\Lambda \cap S^\circ$ we may adjoin all the elements q/t for $q \in Q$ to S°: we need only do this for generators of Q, and so this is still finitely generated over Λ_0. In fact, one has $(S')^\circ = (S^\circ)'$, since $S^\circ = W^{-1}S$ with $W = T - tT$, and t divides s/w in Λ iff t divides $s \in S$. Thus, $t\Lambda \cap S^\circ$ is the expansion of $t\Lambda \cap S$. In fact, it will be convenient in the remainder of the argument to generalize substantially further: for example, we may relax the condition that S be finitely generated over Λ_0 by allowing arbitrary localizations at multiplicative systems disjoint from Q, especially at $S - Q$ itself. We shall also relax the condition that the map $S \to \Lambda$ be injective.

Néron Desingularization

We therefore start over in a new, abstract situation which captures what we need from the original situation. Let $(\Lambda_0, t\Lambda_0, L_0) \hookrightarrow (\Lambda, t\Lambda, L)$ be a local injection of discrete valuation rings such that induced map of fraction fields $\mathcal{F} \to \mathcal{G}$ is separable, and $L_0 \to L$ is separable. Let S be a localization of a finitely generated Λ_0-algebra, torsion-free over Λ_0, and suppose that we have a map $\phi : S \to \Lambda$, but now, instead of assuming that $S \to \Lambda$ is injective, we assume that the kernel is a minimal prime $\mathfrak{q}$ of S such that $S_\mathfrak{q}$ is a field. Then $S_\mathfrak{q}$ is a subfield of $\mathcal{G}$ finitely generated over $\mathcal{F}$.

We define $\Omega'_S = \Lambda \otimes \Omega_{S/\Lambda_0}$. We let ℓ_S denote the length of the torsion-submodule of Ω'_S. If we represent S as $W^{-1}\Lambda_0[Y_1, \ldots, Y_d]/P$ and choose generators $F_1, \ldots, F_k$ for P in $\Lambda_0[Y_1, \ldots, Y_d]$, then if $\mathcal{J}'$ denotes the image of the Jacobian matrix $\left(\dfrac{\partial F_j}{\partial Y_i}\right)$ in Λ, we have that the rank of $\mathcal{J}'$ is the same as the height of the minimal prime of P which is the inverse image of $\mathfrak{q}$ in $W^{-1}\Lambda_0[Y_1, \ldots, Y_d]$. We denote this height by h and we let ℓ_S be the length of torsion submodule of Ω'_S. This length is the same as the minimum order with respect to t of any size h minor of $\mathcal{J}'$, where the minor is viewed as an element of Λ. By enlarging W we can get P itself to be prime, and we can do this while only localizing at elements not divisible by t. As earlier, we may let Q be the contraction of $t\Lambda$ to S and we may let

$S' = S[q/t : q \in Q] \subseteq S_t$, which is finitely generated over S. This algebra is called *Néron's blowing-up of* S. The constructions of both S' and ℓ_S from S commute with localization at a multiplicative system disjoint from Q. Note that we still have a map $S' \to \Lambda$: its value on q/t for $q \in Q$ is $\phi(q)/t$, which is an element of Λ because, by definition of Q, $\phi(q) \in t\Lambda$.

We can now state and, eventually, prove a result more general than the Theorem stated last time. Before proving this, we note that if the field L' is separable over the field L and $u_1, \ldots, u_d$ is a set of field generators, then a separating transcendence basis may be chosen from the elements $u_1, \ldots, u_d$. To see this, consider the proof of (3) $\Rightarrow$ (4) in the first Proposition in the Lecture Notes from November 19. Choose a transcendence basis from $u_1, \ldots, u_d$, say $u_1, \ldots, u_s$, so as to minimize $[L' : L(u_1, \ldots, u_s)^{\mathrm{sep}}]$. If

$$L' \neq L(u_1, \ldots, u_s)^{\mathrm{sep}},$$

choose an element

$$v \in L' - L(u_1, \ldots, u_s)^{\mathrm{sep}}$$

from among the elements $u_{s+1}, \ldots, u_d$. The rest of the argument producing a contradiction is the same.

Theorem. *With notation as above, so that $L_0 \to L$ and $\mathcal{F} \to \mathcal{G}$ are both separable, $\ell_{S'} \leq \ell_S$, with equality if and only if $\ell_S = 0$.*

Proof. We replace S by S_Q. Henceforth we may assume that (S, Q) is local, and that it is a localization of $\Lambda_0[s_1, \ldots, s_d]$. Note that we have a presentation of S of the form

$$S \cong W^{-1}\Lambda_0[Y_1, \ldots, Y_d]/(F_1, \ldots, F_k).$$

We know that at least one size h minor of $\mathcal{J}'$ has minimum order ℓ_S. We are free to make local étale extensions Λ_0^* of Λ_0 and Λ^* of Λ to enlarge their residue class fields (we shall say more about this below). This is done in such a way that we still have a map $\Lambda_0^* \to \Lambda^*$. S is replaced by $\Lambda_0^* \otimes_{\Lambda_0} S$. The details of how to do this will be given in the next lecture.

After enlarging the residue class fields, if necessary, we can make an invertible linear change of variables among $Y_1, \ldots, Y_d$ over Λ_0 so that every h rows of the new $\mathcal{J}'$ will have a size h minor of order ℓ_S. Assuming this for the moment, we renumber the Y_j so that the images of $Y_1, \ldots, Y_\tau$ are a separating transcendence basis for S/Q over L_0: the images of Y_j are field generators, and so we can do this by the remarks prior to the statement of the Theorem. We now replace Λ_0 by

$$\Lambda_1 = \Lambda_0[s_1, \ldots, s_\tau]_{(t)}.$$

We need to understand what happens to ℓ_S and $\ell_{S'}$ when we do this. We shall show that we can keep ℓ_S the same while $\ell_{S'}$ can only increase. Again, the details will be given in the next lecture. Next, we adjust the rings Λ_0 and Λ again so that Λ_0 contains representatives of all elements of S/Q, which is a now a finite separable algebraic extension of L_0. Once again, the details will be given in the next lecture.