Math 711: Lecture of December 3, 2004

We next consider certain changes that we can make without affecting the problem.

We can arrange for a finite separable extension of the residue class field of Λ_0: at the same time, we enlarge Λ. Any such separable extension is generated by a single primitive element θ. We may choose a monic minimal polynomial for θ over L_0: it will be a separable polynomial, and we then lift it to a monic polynomial, $f(z)$, of the same degree over Λ_0. Then $\Lambda_0^* = \Lambda_0[z]/\bigl(f(z)\bigr)$ is a module-finite étale extension of Λ_0. Mod (t) we get the desired extension of L_0. Because killing (t) produces a field, Λ_0^* is still a local ring and, in fact, a DVR with maximal ideal $t\Lambda_0^*$. However, this ring no longer maps to Λ. Consider $\Lambda_0^* \otimes_{\Lambda_0} \Lambda$, which is module-finite and étale over Λ: if we kill (t), we get $L[z]/(f)$, an étale extension of L which is necessarily a product of finite separable extensions of L. Thus, by localizing $\Lambda_0^* \otimes_{\Lambda_0} \Lambda$ at a minimal prime $\mathcal{Q}$ of t, we obtain a DVR, Λ^*, to which Λ_0^* maps, and its maximal ideal is still generated by t. Moreover, $S^* = \Lambda_0^* \otimes_{\Lambda_0} S$ maps to it as well, and we replace Λ_0, S, Λ and $S \to \Lambda$ by these. Note that $\mathcal{J}'$ does not change, although its entries are now considered in Λ^*, and the orders of its minors with respect to t don't change.

We also need to see that this process commutes with Néron's blowing-up. We are assuming that $S = S_Q$, so that S/Q is a field. Note that we might as well replace S^* by $S^*_{Q^*}$, where Q^* is the contraction of $t\Lambda^*$ to S^*. The key point is that the expansion of Q to $S^*_{Q^*}$ is $Q^*S^*_{Q^*}$. Thus, if $q_1, \ldots, q_k$ generate Q, their images generate $Q^*S^*_{Q^*}$, and so Néron's blowing-up of the latter is generated over it by the images of $q_1/t, \ldots, q_k/t$. To see this, it suffices to see that $S^*_{Q^*}/QS^*_{Q^*}$ is a field. But this ring is $(S^*/QS^*)_{Q^*}$, and $S^*/QS^* = \Lambda_0^* \otimes_{\Lambda_0} S/Q$, where S/Q is a field. This is an étale extension of S/Q, and so it is a finite product $\mathcal{F}_1 \times \cdots \times \mathcal{F}_j$ of finite separable algebraic field extensions of S/Q. Since $(S^*/QS^*)_{Q^*}$ is a local ring which is a localization of $S^*/QS^* \cong \mathcal{F}_1 \times \cdots \times \mathcal{F}_j$, it is isomorphic to some $\mathcal{F}_i$, and therefore is a field. It follows that $(S^*_{Q^*})' = (\Lambda_0^* \otimes_{\Lambda_0} S')_{Q^*}$. It then follows that the image of the Jacobian matrix for Néron's blowing-up after we enlarge the residue class field is the same as before, although it is now being considered in Λ^*. The smallest order of a minor with respect to t does not change, however, since $t^i\Lambda^* \cap \Lambda = t^i\Lambda$ for all i.

An invertible linear change of the variables $Y_1, \ldots, Y_d$ over Λ_0 produces a corresponding change in the Jacobian matrix. Suppose that for $1 \le i \le d$, we have that $Y_i = \sum_{k=1}^d \alpha_{ki}Y'_k$, where $\bigl(\alpha_{ki}\bigr)$ is an invertible matrix with entries in Λ_0. Then

$$dY_i = \sum_{k=1}^d \alpha_{ki}dY'_k,$$

and so

$$dF_j = \sum_{i=1}^d \frac{\partial F_j}{\partial Y_i}dY_i = \sum_{i=1}^d \bigl(\frac{\partial F_j}{\partial Y_i}\sum_{k=1}^d \alpha_{ki}dY'_k\bigr) = \sum_{k=1}^d \bigl(\sum_{i=1}^d \alpha_{ki}\frac{\partial F_j}{\partial Y_i}\bigr)dY'_k.$$

Thus, the new Jacobian matrix is $(\alpha_{ki})\mathcal{J}$. We therefore see that an invertible linear change of variables over Λ_0 enables us to perform corresponding row operations on $\mathcal{J}$ and, hence, on $\mathcal{J}'$.

To see that we may assume that any h rows of $\mathcal{J}'$ have a size h minor of order ℓ_S, it suffices to prove the following:

Lemma. *Let $(\Lambda_0, t\Lambda_0, L_0) \subseteq (\Lambda, t\Lambda, L)$ be a local inclusion of discrete valuation rings. Suppose that the residue class field of L_0 is infinite. Let J be a $d \times k$ matrix over Λ of rank h such that the smallest order with respect to t of any size h minor is ℓ. Then one can perform elementary row operations on the matrix over Λ_0 such that any h distinct rows of the matrix have a size h minor of order ℓ.*

Proof. Note that column operations over Λ do not affect the order of the generator of the ideal of h size minors of a given set of h rows. Therefore, we may perform column operations over Λ and row operations over Λ_0 without affecting the issue. First permute rows and columns so that an element of least order is in the upper right hand corner of J. By multiplying the first column by a unit we may assume that the entry in the upper left corner is t^{a_1}. By performing elementary column operations over Λ we may assume that the other entries of the first row are 0. We now iterate this procedure, working with the submatrix obtained by deleting the first row and leftmost column. After h iterations we reach a matrix such that the $h \times h$ submatrix in the upper left corner is lower triangular with $t^{a_1}, \ldots, t^{a_h}$ on the diagonal, the entries to the right of these in the first h rows are 0, and each of the entries in the jth column below t^{a_j} is divisible by t^{a_j}, $1 \le j \le h$. It also follows that the entries in columns beyond the hth column and below the hth row must all be zero, or the rank will be larger than h.

We want to perform elementary row operations over Λ_0 to get every $h \times h$ minor of the first h columns to have order $a_1 + \cdots + a_h$ with respect to t. We factor t^{a_j} from the jth column for $1 \le j \le h$. We may drop all but the first h columns, since the other columns are 0. We thus obtain a matrix with h columns such that the submatrix formed from the first h rows is lower triangular, with 1 in each spot on the main diagonal. It will suffice to perform elementary row operations over Λ_0 so that all size h minors are units.

For this purpose we may work mod $t\Lambda_0$. Thus, we may view the given matrix as having entries in $L = \Lambda/t\Lambda$, and we need only select, for each of the last $d - h$ rows, an L_0-linear combination over Λ_0 of the first h rows to add to it, so as to produce a matrix in which any h rows are independent. This is clearly possible if L_0 is infinite. $\square$

We may now pass to a situation in which any h rows of $\mathcal{J}'$ have an $h \times h$ minor of order ℓ_S. If L_0 is infinite we can do this as in the the Lemma above. If L_0 is finite we can make an étale extension of Λ_0 to enlarge the residue class field enough so that we can make the required linear change of variables.

Then, after renumbering variables, we may assume that the images of $Y_1, \ldots, Y_\tau$ are a separating transcendence basis for S/Q. We now replace Λ_0 by Λ_1, the localization of $\Lambda_0[Y_1, \ldots, Y_\tau]$ at $t\Lambda_0[Y_1, \ldots, Y_\tau]$. S remains the same: the elements we inverted to form Λ_1 are already invertible in $S = S_Q$. Note that ℓ_S does not change: the new Jacobian is computed using only the Y_j for $j > \tau$, and so it is clear that ℓ_S cannot decrease. Moreover, it does not increase, because we have placed the coordinates in general position, and so we can find an $h \times h$ minor of some remaining h rows of order ℓ_S. S' does not change, but the relevant Jacobian matrix is a truncation of the one we used to compute $\ell_{S'}$ before, and so $\ell_{S'}$ can not decrease when replace Λ_0 by Λ_1.

In this way, we may assume that S/Q is a finite separable extension of L_0. But then an étale change of rings for Λ_0 enables us to reduce to the case where all residues in S/Q are represented in L_0. This means that by subtracting scalars in Λ_0 we may assume without loss of generality that all of $s_1, \ldots, s_d \in Q$. This means that the contraction of Q to $\Lambda_0[Y_1, \ldots, Y_d]$ is $(t, Y_1, \ldots, Y_d)$.

The main calculation in the proof of the Theorem. We can now complete the proof that iterating Néron's blowing-up eventually produces an algebra S such that $\ell_S = 0$. We need only show that $\ell_{S'} \leq \ell_S$ with strict inequality if $\ell_S > 0$.

Note that the generators of S' are the elements s_i/t. Let Z_i be variables mapping to these. We do not have to find all the relations on the Z_i: we only need sufficiently many to be able to see that some size h minor of the new Jacobian matrix has the same order as before, and that there exists a minor of smaller order if $\ell_S > 0$.

Let

$$F_j = b_j + \sum_{i=1}^{d} a_{ij} Y_i + H_j(Y)$$

where H_j has only terms of degree 2 and higher. Note that all coefficients are in Λ_0. Since $F_j(s_1, \ldots, s_d) = 0$, we may substitute $Y_i = tZ_i$ to get a polynomial in the Z_i which gives a relation on the s_i/t: it has the form

$$b_j + t \sum_{i=1}^{d} a_{ij} Z_i + t^2 H_j^*(Z),$$

where the coefficients are in Λ_0. Here $b_j \in \Lambda_0 \cap t\Lambda = t\Lambda_0$, and so we may write $b_j = tc_j$, with $c_j \in \Lambda_0$. We get a relation

$$G_j = c_j + \sum_{i=1}^{d} a_{ij} Z_i + tH_j^*(Z).$$

Note that $G_j(Z) = F_j(tZ)/t$, whence

$$\frac{\partial G_j(Z)}{\partial Z_i} = 1/t\big(\frac{\partial F_j}{\partial Y_i}(tZ)\big)t$$

where the final factor t arises from the chain rule. Substituting $Z_i = s_i/t$, we see that the image of the Jacobian matrix $\big(\frac{\partial G_j}{\partial Z_i}\big)$ in Λ is $\mathcal{J}'$. This proves that $\ell_{S'} \leq \ell_S$.

Finally, suppose that $\ell_S > 0$. This means that mod $t\Lambda$, all size h minors of $\mathcal{J}'$ vanish, so that the rank of this matrix mod $t\Lambda$ is at most $h-1$. Mod t, the matrix is the same as the image of $\bigl(a_{ij}\bigr)$ mod t. We may assume, after renumbering, that ℓ_S is achieved in the minor obtained by using the first h columns and the first h rows of $\mathcal{J}'$.

Now , mod $t\Lambda \cap \Lambda_0 = t\Lambda_0$, the the first h columns of $\mathcal{J}'$ are linearly dependent over L_0, and hence so are the first h columns of $\bigl(a_{ij}\bigr)$. By renumbering, we may assume without loss of generality that the first column is an L_0-linear combination of the columns numbered 2 through h. Lift the elements of L_0 needed to elements $\lambda_2, \ldots, \lambda_h \in \Lambda_0$, and consider $G_1 - \lambda_2 G_2 - \cdots - \lambda_h G_h$. The coefficients on quadratic and higher terms are divisible by t. By our choice of the λ_ν, the coefficients on the Z_i are divisible by t. The scalar is then forced to be divisible by t. Thus, $G_1 - \lambda_2 G_2 - \cdots - \lambda_h G_h$ may divided by t to get a new relation, which we call G_0. Consider the Jacobian matrix of $G_0, G_2, \ldots, G_h$ with respect to the variables $Z_1, \ldots, Z_h$, and compare this with the Jacobian matrix of $G_1, \ldots, G_h$ with respect to the variables $Z_1, \ldots, Z_h$. The first column has been altered first by subtracting off a sum of multiples of the other columns, which does not affect the minor, and then by factoring out t from the first column, while the other columns are the same as when we used for $G_1, \ldots, G_h$. The new minor clearly has order $\ell(S) - 1$ with respect to t. $\square$

This completes not only the proof of the Theorem asserted in the previous lecture, but also the proof of all cases of the Artin Approximation Theorem.

We now want to use Artin Approximation to prove the following:

Theorem. *Consider a family of finite systems of polynomial equations over $\mathbb{Z}$ such that each system in the family involves variables $x_1, \ldots, x_d$ and other variables $Y_{i,1}, \ldots, Y_{i,h_i}$ where both h_i and the variables are allowed to depend on which system in the family one is considering. Suppose that none of these systems has either*

(a) *a solution in a finitely generated algebra over a finite field such that the values of the x_j generated an ideal of height d, nor*

(b) *a solution in a finitely generated algebra over a DVR of mixed characteristic $p > 0$ such that the ideal generated by the values of the x_j has height d.*

Then no system in the family has a solution in a Noetherian ring in such a way that the values of the x_j generate an ideal of height d. Moreover, (a) alone guarantees that there is no solution in a Noetherian ring containing a field such the values of the x_j generate an ideal of height d.