Math 711: Lecture of December 8, 2004

Lemma. *Let R be an algebra finitely generated over its prime ring. Then the quotient of R by any maximal ideal is a finite field.*

Proof. The quotient S will be a field finitely generated as an algebra over its prime ring $\mathbb{Z}$ or $\mathbb{Z}/p\mathbb{Z}$. If the prime ring is $\mathbb{Z}$ then, by Noether normalization for domains, after localizing at one nonzero element of $\mathbb{Z}$, S is a module-finite extension of a polynomial extension by finitely many variables of $\mathbb{Z}_a$, $a \neq 0$. If the prime ring is $\mathbb{Z}/p\mathbb{Z}$, S is a module-finite extension of a polynomial extension of $\mathbb{Z}/p\mathbb{Z}$. Since S is a field, it has dimension 0, which is impossible if the prime ring is $\mathbb{Z}$, since module-finite extensions preserve dimension, and adjoining indeterminates increases dimension by the number of indeterminates adjoined. In the second case one sees that there are no indeterminates, and S is module-finite over $\mathbb{Z}/p\mathbb{Z}$, which means that it is a finite field. $\square$

Lemma (Generic Freeness). *Let M be a finitely generated module over R, where R is a finitely generated algebra over the Noetherian domain A. Then there is a nonzero element $a \in A$ such that M_a is A_a-free.*

Proof. M has a prime cyclic filtration by modules of the form R/P, and it suffices to show this for R/P, which can replace R. So it suffices to do the case $M = R$ and R is a domain. We use induction on $\dim(\mathcal{F} \otimes_A R)$, where $\mathcal{F}$ is the fraction field of A. By Noether normalization R is module-finite over a $A_a[X_1, \ldots, X_n]$, and so it suffices to consider a prime cyclic filtration of R_a over $A_a[X_1, \ldots, X_n]$. Those factors that are equal to $A_a[X_1, \ldots, X_n]$ are free, those that have A_a-torsion become zero after localization at one more element of $A - \{0\}$, while the other factors can be made free by localizing at one more element of $A - \{0\}$ by the induction hypothesis. $\square$

In the situation we were considering last time, we now know that every B/μ, for μ maximal in B, is a finite field. We want to preserve the condition that the $x_1, \ldots, x_d$ generate an ideal whose radical is a maximal ideal of height d, and this is equivalent to saying that the radical is maximal of height $\geq d$. Note that the height is correct when we pass to the fraction field $\mathcal{L}$ of B, since after that we are making a base change from one field to another, and the height will be preserved by the following much more general result:

Proposition. *Let S be a Noetherian ring faithfully flat over R and let I be an ideal of R. Then the height of IS is the same as the height of I. In particular, if R is a finitely generated $\mathcal{L}$-algebra and K is a field containing $\mathcal{L}$, then for every ideal I of R, the height of I is the same as the height of its expansion to $K \otimes_{\mathcal{L}} R$.*

Proof. The height of IS is the same as the minimum of the heights of minimal primes Q of S containing IS. If we replace S by S_Q for such a prime Q and R by its localization at the contraction of Q, then $R_P \to S_Q$ is faithfully flat. IS_Q is primary to QS_Q, and it follows that PR_P is nilpotent modulo IR_P. Then the height of Q is $\dim(S_Q) \geq \dim(R_P)$ which is at least the height of I. Now choose P prime in R so that it is a minimal prime of I

and the height of I is $\dim(R_P)$. Choose Q to be a minimal prime of PS. Then $R_P \to S_Q$ is faithfully flat with closed fiber of dimension 0, and so we have that the height of IS is bounded by $\dim(S_Q) = \dim(R_P)$ (see part (d) of the first Lemma of the Lecture Notes of October 11) which is the height of I. The second statement follows from the first because K is faithfully flat over $\mathcal{L}$ and so $K \otimes_{\mathcal{L}} R$ is faithfully flat over R. $\square$

We are free to replace B by its localization at one nonzero element several (but finitely many) times: we shall retain the notation B as we do this.

Moreover, we are free to localize at one (equivalently, finitely many) nonzero elements of B: this is equivalent to looking at a dense open subset of MaxSpec (B). In $R_{\mathcal{L}} = \mathcal{L} \otimes_B R_B$ we can chooae a minimal prime contained in the radical of $(x_1, \ldots, x_d)$ so as to preserve the height when this prime is killed. We may kill the contraction of this prime to R_B, and so assume that R_B and $R_{\mathcal{L}}$ are domains. After localization at one element of $B - \{0\}$ we may assume that R_B is module-finite over a polynomial $B[u_1, \ldots, u_d]$, and embeds in a finitely generated free module G_B over $B[u_1, \ldots, u_d]$. After localizing at one more element of $B - \{0\}$ we may assume that G_B/R_B is free over B, by the Lemma on generic freeness. Thus, for any μ, $R_{B/\mu}$ is module-finite over $(B/\mu)[u_1, \ldots, u_d]$ and embeddable in a torsion-free module over it: thw sequence $0 \to R_B \to G_B \to G_B/R_B \to 0$ remains exacts when we apply $\_ \otimes_B B/\mu$ because G_B/R_B is B-free and so $\mathrm{Tor}_1^B(G_B/R_B, B/\mu) = 0$. This implies that it has pure dimension d, and so every maximal ideal has height d. Now $\mathcal{L} \otimes_B R_B/(x_1, \ldots, x_d)$ is local and Artinian, with a nilpotent maximal ideal. The residue field is module-finite over $\mathcal{L}$. Therefore, after suitable localization, we may assume that $R_B/(x_1, \ldots, x_d)$ has a nilpotent prime such that the quotient is a domain module-finite over B. Now it is clear that all maximal ideals of $R_{B/\mu}$ have height d, and that the quotient of this ring by $(x_1, \ldots, x_d)$ is 0-dimensional, which forces $(x_1, \ldots, x_d)$ to have height d, since all of its minimal primes must be maximal ideals.

If there are several minimal primes among these maximal ideals, we can get back to the case where there is just one by localizing at one element. $\square$

We want to apply our method of reduction to characteristic p to prove that every local ring containing a field has a big Cohen-Macaulay module. A module M over a local ring (R, m, K) is called a *big Cohen-Macaulay module* for R if every system of parameters for R is a regular sequence on M and $mM \neq M$.

Theorem. *Every local ring that contains a field has a big Cohen-Macaulay module.*

We shall prove this by showing that the existence of a big Cohen-Macaulay module is an equational problem, and then it will suffice to solve the problem in positive characteristic. Stronger results are known: e.g., it is known that there are big Cohen-Macaulay algebras for local rings that contain a field and in mixed characteristic in dimension at most three. The proofs of these stronger results are extremely difficult.

The result of the Theorem above is very useful, and will illustrate the method of reduction to characteristic p. For rings of equal characteristic zero, no proof of the result is known without reduction to characteristic $p > 0$.