Math 711: Lecture of December 10, 2004

In order to construct big Cohen-Macaulay modules over a local ring $(R,\, m,\, K)$, we want to discuss the notion of a modification of a module M with respect to a set of relations in M with respect to sequences $x_1,\, \ldots\,, x_{k+1}$ that are part of a system of parameters for the local ring. The careful study of this idea will enable us to see that the existence of big Cohen-Macaulay modules is equivalent to the statement that a certain family of polynomial equations with a dimensional constraint has no solution.

Suppose that $\dim(R) = d$. Fix a non-empty family $\mathcal{F}$ of sequences of length d such that the elements of each sequence form a system of parameters for R. An important special case is when $\mathcal{F}$ has just one element. Another is when $\mathcal{F}$ consists of all such sequences.

We shall modify our terminology a bit and say that an R-module M is a *big Cohen-Macaulay module* with respect to $\mathcal{F}$ if every sequence in $\mathcal{F}$ is a regular sequence on M. (Thus, our original use of the term *big Cohen-Macaulay module* corresponds to the case where $\mathcal{F}$ consists of all sequences of parameters.) By our definition of regular sequence, for $x_1,\, \ldots\,, x_n$ to be a regular sequence on M, we must have that $(x_1,\, \ldots\,, x_n)M \neq M$. For any ideal generated I generated by a system of parameters (or any m-primary ideal I), the condition that $IM \neq M$ is equivalent to the condition that $mM \neq M$. For if $mM = M$, we have that $m^2M = m(mM) = mM = M$, and, by induction, that $m^tM = M$ for all t. Since $m^t \subseteq I$ for some t, we have that $IM = M$. On the other hand, if $IM = M$ it is clear that $mM = M$.

By a type k-relation on a module M with respect to $\mathcal{F}$ we mean a sequence of elements $(u_1,\, \ldots\,, u_k, u)$ in M together with a sequence $x_1,\, \ldots\,, x_{k+1}$ that is an initial segment of an element of $\mathcal{F}$ such that

$$x_{k+1}u = \sum_{i=1}^{k} x_i u_i$$

(the case $k = 0$ is allowed, and then the right hand side is 0).

Given a set of relations $\mathcal{S}$ with respect to $\mathcal{F}$ (the types are allowed to vary), by the *modification* of M with respect to $\mathcal{S}$ we mean the map $M \to M'$ constructed as follows. For each element $\sigma \in \mathcal{S}$ of type $k = k_\sigma$, let G_σ be a free R-module of rank k with free basis $b_1^\sigma,\, \ldots\,,\, b_k^\sigma$, and let

$$M' = (M \oplus \bigoplus_{\sigma \in \mathcal{S}} G_\sigma)/\mathrm{Span}_R\{\rho_\sigma : \sigma \in \mathcal{S}\}$$

where, if σ corresponds to

$$x_{k+1}u = \sum_{i=1}^{k} x_i u_i,$$

then

$$\rho_\sigma = u - \sum_{i=1}^{k} x_i b_i^\sigma.$$

The map $M \to M'$ is the composition of the obvious injection of M into the direct sum in the numerator composed with the quotient surjection.

If the set $\mathcal{S}$ has just one element, we shall say that M' is a *single modification* of M.

Given a modification of M with respect to a set of relations, call it M', one can then form a modification of M', call it M'', with respect to some set of relations on M''. Continuing in this way, one may consider sequences of modifications

$$M \to M' \to M'' \to \cdots \to M^{(r)}.$$

We start with R itself, and form a sequence in which, at each stage, we modify the given module with respect to the set of *all* relations on M with respect o $\mathcal{F}$. In this way we get a sequence of modifications

$$R = M_0 \to M_1 \to M_2 \to \cdots.$$

Each relation with respect to $\mathcal{F}$ on M_i becomes trivial in M_{i+1}. Let M_∞ be the direct limit of the M_i. The image of $1 \in R$ in M_i plays a special role here: call it 1_i. In particular, the image of 1 in M_∞ is denote 1_∞.

Lemma. *Let $(R,\, m,\, K)$ of dimension d be given, let $\mathcal{F}$ be non-empty family of sequences of length d whose elements are systems of parameters, as above, let $x_1, \ldots, x_d$ be one element of $\mathcal{F}$ and let*

$$R = M_0 \to M_1 \to \cdots \to M_r \to \cdots$$

by the sequence of modifications with respect to $\mathcal{F}$ described in the preceding paragraph. Let M_∞ be the direct limit of the M_r, and let 1_r be the image of $1 \in R$ in M_r, $0 \le r \le \infty$. Then the following conditions are equivalent:

(1) *R has a big Cohen-Macaulay module with respect to $\mathcal{F}$.*

(2) *M_∞ is a big Cohen-Macaulay module over R with respect to $\mathcal{F}$.*

(3) $1_\infty \notin (x_1, \ldots, x_d)M_\infty$.

(4) *$1_r \notin (x_1, \ldots, x_d)M_r$ for all positive integers r.*

(5) *For every sequence $M_0 = R, \ldots, M_r$ such that each M_{i+1} is a modification of M_i with respect to some set of relations on M_i over $\mathcal{F}$, the image of $1 \in R$ in M_r is not in $(x_1, \ldots, x_d)M_r$.*

Moreover, if M is any R-module with a map ϕ to a big Cohen-Macaulay module B with respect to $\mathcal{F}$, and M' is any modification of M with respect to a set of relations on M over $\mathcal{F}$, then the map $M \to B$ factors through a map $M' \to B$, so that one has $M \to M' \to B$.

Proof. It is clear that (5) $\Rightarrow$ (4), while the equivalence of (3) and (4) follows from the fact that M_∞ is the direct limit of the M_r. If (3) holds, we claim that (2) holds. Evidently, with $I = (x_1, \ldots, x_d)R$ we have $IM \neq M$, and therefore, by the discussion at the end of the third paragraph of the first page, we have that $mM \neq M$. But given a relation on $x_1, \ldots, x_{k+1}$ in M_∞, where $x_1, \ldots, x_{k+1}$ is an initial segment of some sequence in $\mathcal{F}$, it must come from such a relation on some M_r that maps to it. This relation becomes trivial in M_{r+1}, and therefore in M_∞ as well. Of course, (2) $\Rightarrow$ (1) is clear.

It remains only to prove that (1) $\Rightarrow$ (5). It will suffice to show that given a big Cohen-Macaulay module B and an element $w \in B - mB$, we can map $M_r \to B$ in such a way that the image w_r of $1 \in R$ in M_r maps to w. We cannot then have $w_r \in (x_1, \ldots, x_d)M_r$, for we may apply the map $M_r \to B$ then gives that $w \in (x_1, \ldots, x_d)B \subseteq mB$, a contradiction.

We show that there is a map $M_r \to B$ by defining it successively on the sequence of modules $R = M_0, M_1, \ldots, M_r, \ldots$. In the case of R, we simply take the map that sends $1 \in R$ to $w \in B$. We use induction. Suppose that we have defined $\phi_i : M_i \to B$ such that $\phi_i(w_i) = w$, where w_i is the image of 1 in M_i. We want to define $\phi_{i+1} : M_{i+1} \to B$ such that the diagram

$$\begin{tikzcd} B \arrow[r, "="] & B \\ M_i \arrow[u, "\phi_i"] \arrow[r] & M_{i+1} \arrow[u, "\phi_{i+1}"'] \end{tikzcd}$$

commutes. That is, we want to establish the final statement of the Lemma with $M = M_i$, $\phi = \phi_i$ and $M' = M_{i+1}$, and so we can complete the proof by proving the final statement of the Lemma.

We proceed by defining the new map ϕ' on each direct summand of $M \oplus \bigoplus_\sigma G_\sigma$ so as to kill all the relations ρ_σ. We define it to be ϕ on M. We need to specify values for ϕ' on the free generators of every G_σ in such a way that all the ρ_σ vanish. Given σ corresponding to $x_{k+1}u = \sum_{j=1}^k x_i u_i$ with the u_i and u in M, we apply ϕ to get

$$x_{k+1}\phi(u) = \sum_{j=1}^{k} x_i \phi(u_i).$$

Since B is a big Cohen-Macaulay module with respect to $\mathcal{F}$, it follows that $\phi(u) \in (x_1, \ldots, x_k)B$ which yields $\phi(u) = \sum_{j=1}^k x_i v_i$ for suitable elements $v_1, \ldots, v_k \in B$. We define the values of ϕ' on $b_1^\sigma, \ldots, b_k^\sigma$ to be $v_1, \ldots, v_k$, respectively. This clearly does what we need. $\square$

Remark. In conditions (3), (4), and (5) we could use mM instead of $(x_1, \ldots, x_d)M$. The given formulation is convenient when we formulate an equational version of the criterion.

Note that if $f : M \to N$ is any R-linear map, each type k relation on M with respect to $\mathcal{F}$ maps to a type k-relation on N with respect to $\mathcal{F}$: the point is simply that if

$$x_{k+1}u = \sum_{i=1}^{k} x_i u_i$$

then

$$x_{k+1}f(u) = \sum_{i=1}^{k} x_i f(u_i).$$

In particular, if one has a sequence of modifications of M, each type k relation on M maps to such a relation on the further terms in the sequence.

Our next objective is to show that the obstruction to the existence of big Cohen-Macaulay modules with respect to $\mathcal{F}$ can be phrased in terms of (all) finite sequences of single modifications of M. We make use of characterization (4). Suppose that $1_r \in (x_1, \dots, x_d)M_r$. Only finitely many elements of M_r are needed as coefficients here. The same relation will hold if we modify M_{r-1} with respect to only finitely many of the relations used in the construction of M_r. All of these modifications can be described using only finitely many elements of M_{r-1}. All the elements and relations needed will be in a modification of M_{r-2} with respect to finitely many relations. We can continue working backward in this way. We therefore get the following:

Theorem. *Let (R, m, K) be local of dimension d, let $\mathcal{F}$ be a non-empty family of sequences of length d whose elements are systems of parameters for R, and let $x_1, \dots, x_d$ be one element of $\mathcal{F}$. The following conditions are equivalent:*

(1) *R has a big Cohen-Macaulay module with respect to $\mathcal{F}$.*

(2) *For every finite sequence of modifications of R, say*

$$R = M_0 \to M_1 \to \cdots \to M_r,$$

each with respect to a finite set of relations over $\mathcal{F}$, the image w_r of $1 \in R$ in M_r is not in $(x_1, \dots, x_d)M_r$.

(3) *For every finite sequence of single modifications of R, say*

$$R = M_0 \to M_1 \to \cdots \to M_r,$$

each with respect to a single relation over $\mathcal{F}$, the image w_r of $1 \in R$ in M_r is not in $(x_1, \dots, x_d)M_r$.

Proof. The argument prior to the statement of the Theorem shows that if (2) holds then condition (4) of the Lemma holds, and so (2) $\Rightarrow$ (1). However, a modification of a module M with respect to finitely many relations $\sigma_1, \dots, \sigma_h$ can be achieved by making, successively, h single modifications: one modifies M with respect to σ_1 to get M_1, and then modifies M_1 with respect to the image of σ_2, and so forth. At the inductive step, one modifies M_i with respect to the image of σ_{i+1}, if $i < h$. Then M_h is the same as the modification of M with respect to $\sigma_1, \dots, \sigma_h$. Thus, (3) $\Rightarrow$ (2). Finally, (1) $\Rightarrow$ (3) follows from the implication (1) $\Rightarrow$ (5) in the preceding Lemma. $\square$

We next want to show that the problem of the existence of big Cohen-Macaulay modules can be viewed equationally. For simplicity, we first state the next result when there is only one system of parameters being used, and then indicated the modification needed when there may be many.

Theorem. *Suppose that (R, m, K) is a d-dimensional local ring with system of parameters $x_1, \dots, x_d$. Let $k_1, \dots, k_r$ be a sequence of integers with values between 0 and*

$d-1$. *Then there is a system of polynomial equations with coefficients in* $\mathbb{Z}$ *and variables* $X_1, \ldots, X_d, Y_1, \ldots, Y_h$ *such that the system has a solution in* R *with* $x_1, \ldots, x_d$ *as the values of* $X_1, \ldots, X_d$ *if and only if* R *has a sequence of single modifications*

$$R = M_0 \to M_1 \to \cdots \to M_r$$

of types $k_1, \ldots, k_r$ *with respect to* $x_1, \ldots, x_d$ *such that the image* w_r *of* 1 *is in the submodule* $(x_1, \ldots, x_d)M_r$.

Before discussing the proof of this result in the general case, we want to discuss the case where $r = 1$. We write $k = k_1$. The modification comes from a relation

$$yx_{k+1} = \sum_{j=1}^{k} y_k x_k$$

over R. The modification may be described as $(R \oplus R^k)/R\rho$ where

$$\rho = (y, -x_1, \ldots, -x_k).$$

We are then concerned with whether, in this modified module M_1, we have that the image of $(1, 0, \ldots, 0)$ is in $(x_1, \ldots, x_d)M_1$. This leads to the additional equations coming from the vector equation

$$(1, 0, \ldots, 0) = (y_{11}x_1 + \cdots + y_{1d}x_d, \cdots, y_{k+1,1}x_1 + \cdots + y_{k+1,d}x_d) + y'\rho$$

This will give $k+1$ equations over $\mathbb{Z}$ in variables $Y, Y_1, \ldots, Y_k$, $Y_{\mu,\nu}$, Y' and $X_1, \ldots, X_d$, where each lower case letter has been replaced by a correspondingly subscripted or superscripted upper case letter that is to be viewed as a variable.

In the general case, we give explicit presentations of each of $M_1, \cdots, M_r$. These are constructed recursively. M_i will be a quotient of $R \oplus R^{k_1} \oplus R^{k_i} \cong R^{1+k_1+\cdots k_i}$ by the span over R of i vectors, $\rho_1, \ldots, \rho_i$. The ρ_i will be thought of as having entries some unkonwn, and others satisfying certain equations over $\mathbb{Z}$ coming from relations on previous modules in the sequence. To get the presentation of M_{i+1}, one starts with a relation on M_i. This is given by using unknown coefficients: the relation is thought of as being given by vectors in the numerator, which is free, and one adds into the equation a linear combination of the vectors $\rho_1, \ldots, \rho_i$ with unknown coefficients. One adds a copy of $R^{k_{i+1}}$ in the numerator, and identifies $R^{1+k_1+\cdots k_i}$ with its image in $R^{1+k_1+\cdots k_i+k_{i+1}}$ (spanned by an initial segment of the standard basis). The vectors $\rho_1, \ldots, \rho_i$ one killed to get M_i may be identified with their images (one enters zeros in the last k_{i+1} spots). In addition, to get the presentation of M_{i+1} one kills one additional vector, ρ_{i+1}, derived from a new relation holding in M_i, whose entries are unknowns satsifying certain equations.

Eventually one constructs the presentation of M_r, and then one can express the condition that the image of $1 \in R$ is in $(x_1, \ldots, x_d)M_r$ by one additional vector equation, setting $(1, 0, \ldots, 0)$ equal to the sum of a vector whose entries are unknown linear combinations of $x_1, \ldots, x_d$ and a linear combination of the $\rho_1, \ldots, \rho_r$ with unknown coefficients. $\square$