Math 711: Lecture of December 13, 2004

We next want to describe how the equational set-up changes if there is a family of systems of parameters and one has a modification with respect to a system that may be different at every stage. It is convenient to view one of the systems, $x_1, \ldots, x_d$ as special. This one is used when one writes down an equation corresponding to the fact that the image of $1 \in R$ is in $(x_1, \ldots, x_d)M_r$. In dealing with a finite sequence of single modifications, only finitely systems will occur. One can introduce variables that represent the elements in the finitely many systems of parameters. The equations connected with the modification process change only in obvious ways: in dealing with a modification with respect to a certain system of parameters or a relation on a certain system of parameters; one uses those parameters as coefficients (or variables corresponding to them in the system of equations). One introduces extra equations that keep track of the fact that each of the additional sequences has the same radical as $x_1, \ldots, x_d$. If $x'_1, \ldots, x'_d$ is one such sequence, it suffices to use the equations

$$x_j^N = \sum_{i=1}^{d} z_{ij} x'_i$$

(which will hold for some positive integer N and suitable elements $z_{ij} \in R$) and similar ones reversing the roles of the x_i and the x'_i: these equations guarantee that $(x_1, \ldots, x_d)$ and $(x'_1, \ldots, x'_d)$ have the same radical, so that $x'_1, \ldots, x'_d$ is also a system of parameters.

The problem of the existence of big Cohen-Macaulay modules has now been shown to be equational with dimension constraint in a sense that reduces the problem in equal characteristic to the case of local ring in characteristic $p > 0$. But we may then pass to the case of a complete local domain, and it will suffice to prove that case.

We next observe the following fact:

Theorem. *Let R be a complete local domain of positive characteristic p and let $x_1, \ldots, x_d$ be a system of parameters. Then there exists a nonzero element $c \in R$ such that for all k, $0 \le k < d$, and for all $q = p^e$, $e \in \mathbb{N}$,*

$$(*) \quad c\big((x_1^q, \ldots, x_k^q)R :_R x_{k+1}^q\big) \subseteq (x_1^q, \ldots, x_k^q)R.$$

Hence, for any finite set of systems of parameters there exists a nonzero element c that has this property for all of these sets.

Moreover, if R is module-finite over a regular local ring A we can choose $c \in A - \{0\}$ so that it has this property for every system of parameters in A.

Proof. Given one system of parameters $x_1, \ldots, x_d$ we can choose a coefficient field K for R and let $A = K[[x_1, \ldots, x_d]]$. Since systems of parameters for A are closed under the operation of replacing every element by a power of that element, the statement for one system of parameters follows from the statement about A. But if one has several systems of parameters and elements $c_1, \ldots, c_k$, where for all of the systems of parameters one of the

c_i satisfies the condition $(*)$ for that system, then $c = c_1 \cdots c_k$ satisfies the condition $(*)$ for all of them. It therefore suffices to prove the final statement for systems of parameters in A.

Let h denote the torsion-free rank of R as an A-module and let $u_1, \dots, u_h$ be a maximal set of elements in R linearly independent over A. Let G be the A-span of these elements. Then $G \cong A^{\oplus h}$, and R/G is A-torsion. Thus, we may choose $c \in A - \{0\}$ such that c kills R/G, i.e., such that $cR \subseteq G$.

Let $y_1, \dots, y_d$ be any system of parameters for A and suppose that

$$y_{k+1}r = \sum_{i=1}^{k} y_i r_i.$$

Multiply by c to obtain

$$y_{k+1}(cr) = \sum_{i=1}^{k} y_i (cr_i).$$

All of the elements cr, $cr_i \in G$. Since A is regular, it is Cohen-Macaulay, and $y_1, \dots, y_d$ is a regular sequence on A and, therefore, on $G \cong A^{\oplus h}$ as well. It follows that

$$cr \in (y_1, \dots, y_k)G \subseteq (y_1, \dots, y_k)R. \quad \square$$

We next note the following: let $(R, m) \to (S, n)$ be a local homomorphism of local rings of the same dimension such that mS is primary to n. This implies that the image of every system of parameters for R is a system of parameters for S. Suppose that $x_1, \dots, x_d$ has image $y_1, \dots, y_d$ in S. Let $M \to M'$ be a single modification of M with respect to a relation

$$x_{k+1}u = \sum_{i=1}^{k} x_i u_i,$$

where the $u_i \in M$. Then $S \otimes_R M \to S \otimes_R M'$ is likewise a single modification of $S \otimes_R M$ with respect to the relation

$$y_{k+1}(1 \otimes u) = \sum_{i=1}^{k} y_i (1 \otimes u_i).$$

The verification is quite straightforward:

$$S \otimes_R (M \oplus R^{\oplus k}) \cong (S \otimes_R M) \oplus S^{\oplus k},$$

and the image of $(u, -x_1, \dots, -x_k)$ is $(1 \otimes u, -y_1, \dots, -y_k)$. Thus, a sequence of single modifications

$$R = M_0 \to M_1 \to \cdots \to M_r$$

becomes another sequence of single modifications

$$S = S \otimes_R M_0 \to S \otimes_R M_1 \to \cdots \to S \otimes_R M_r.$$

Moreover, if the image of $1 \in R = M_0$ in M_r is in $(x_1, \ldots, x_d)M_r$, then the image of $1 \in S = S \otimes_R M_0$ in $S \otimes_R M_r$ is in $(y_1, \ldots, y_d)(S \otimes_R M_r)$.

In particular, we can apply base change when $S = R$ and the map is a power of the Frobenius endomorphism, sending $r \mapsto r^q$ for all $r \in R$: here $q = p^e$ for some $e \in \mathbb{N}$.

We are now ready to prove:

Theorem. *Every local ring containing a field has a big Cohen-Macaulay module (with respect to all systems of parameters).*

Proof. As already noted, the problem is equational with dimension constraint and so reduces to the case of a complete local domain (R, m, K) of characteristic $p > 0$. Suppose that there is a series of single modifications with respect to various systems of parameters

$$R = M_0 \to M_1 \to \cdots \to M_r$$

such that the image of $1 \in M_0$ in M_r is in $(x_1, \ldots, x_d)M_r$, where $x_1, \ldots, x_d$ is a system of parameters. We shall obtain a contradiction. Choose $c \in R - \{0\}$ such that condition $(*)$ of the preceding Theorem holds for all of the finitely many systems of parameters that occur in the displayed sequence of modifications. Then for every $q = p^e$ we may apply base change using the eth power of the Frobenius endomorphism, and so obtain a new sequence of modifications

$$R = M_0^{(e)} \to M_1^{(e)} \to \cdots \to M_r^{(e)}$$

such that the image of $1 \in M_0$ in $M_r^{(e)}$ is in $(x_1^q, \ldots, x_d^q)M_r^{(e)}$. Each modification is with respect to a system of parameters consisting of qth powers of the elements in one of the original systems of parameters. We claim that there is is a commutative diagram:

$$\begin{array}{ccccccccccccc}
R & \xrightarrow{c} & R & \xrightarrow{c} & \cdots & \xrightarrow{c} & R & \xrightarrow{c} & R & \xrightarrow{c} & \cdots & \xrightarrow{c} & R \\
{\scriptstyle id_R}\uparrow & & {\scriptstyle \phi_1}\uparrow & & & & {\scriptstyle \phi_i}\uparrow & & {\scriptstyle \phi_{i+1}}\uparrow & & & & {\scriptstyle \phi_r}\uparrow \\
R = M_0^{(e)} & \longrightarrow & M_1^{(e)} & \longrightarrow & \cdots & \longrightarrow & M_i^{(e)} & \longrightarrow & M_{i+1}^{(e)} & \longrightarrow & \cdots & \longrightarrow & M_r^{(e)}
\end{array}$$

where we shall show recursively that the vertical arrows can be constructed so that the diagram commutes. The leftmost arrow is simply the identity map on R.

Suppose that we have constructed the ϕ_j, $j \le i$, so that the leftmost i squares commute, and suppose that $i < r$. Let $M = M_i^{(e)}$, and write $\phi = \phi_i$. We write $M' = M_{i+1}^{(e)}$. Then M' has the form

$$\bigl(M \oplus (Rb_1 \oplus \cdots \oplus Rb_k)\bigr)/Rv$$

where the b_i give a free basis, where

$$v = u - y_1 b_1 - \cdots - y_k b_k$$

where $y_1, \dots, y_d$ is a system of parameters consisting of qth powers of one of the finitely many systems specified earlier, and where there is a relation

$$y_{k+1} u = \sum_{i=1}^{k} y_i u_i$$

for elements $u, u_1, \dots, u_k \in M$. Then

$$y_{k+1} \phi(u) = \sum_{i=1}^{k} y_i \phi(u_i) \in R,$$

and we therefore have that

$$c\phi(u) = \sum_{i=1}^{k} y_i r_i \in R,$$

because of the special choice of c satisfying $(*)$ for each of the finitely many systems of parameters that occur. But this means that we can define the next map on the numerator module $M \oplus (Rb_1 \oplus \cdots \oplus Rb_k)$ by letting it agree with $c\phi$ on M and by mapping each b_i to r_i: our choice of the r_i is such that v is killed.

Once we have this commutative diagram we can compute the image of $1 \in M_0$ in the rightmost copy of R in the upper row by following two different paths: if we apply the leftmost vertical arrow and then all of the horizontal arrows in the top row, we get $1 \cdot c^r = c^r$. If we apply the horizontal arrows in the bottom row and then ϕ_r, we get the image in R of an element in $(x_1^q, \dots, x_d^q)M_r^{(e)}$, which will be in $(x_1^q, \dots, x_d^q)R$. This shows that for all q,

$$c^r \in (x_1^q, \dots, x_d^q)R \subseteq m^q,$$

and so $c^r \in \bigcap_q m^q = 0$, a contradiction. $\square$

We shall now use this theorem to prove two theorems that do not refer to any non-Noetherian objects.

The next result was proved in characteristic p and in certain characteristic 0 cases by Peskine and Szpiro [C. Peskine and L. Szpiro, *Dimension projective finie et cohomologie locale*, Publ. Math. I.H.E.S. (Paris) **42** (1973) pp. 323–395] (for the intersection theorem) and [______, *Syzygies et multiplicités*, C. R. Acad. Sci. Paris Sér. A **278** (1974) pp. 1421–1424] (for the new intersection theorem) and by Paul Roberts [P. Roberts, *Two applications of dualizing complexes over local rings*, Ann. Sci. Ec. Norm. Sup. **9** (1976) pp. 103–106]. The equicharacteristic case in general was obtained in [M. Hochster, *Topics in the Homological Theory of Modules over Commutative Rings*, CBMS Regional Conference Series No. **24**, AMS, Providence, RI, 1975]; see also [______, *Canonical elements in local cohomology modules and the direct summand conjecture*, J. of Algebra **84** (1983) pp. 503–553].

Theorem (new intersection theorem). *Let R be a local ring that contains a field. Let $0 \to G_n \to \cdots \to G_0 \to 0$ be a finite complex of finitely generated free modules such that $H_0(G_0) \neq 0$ and all the homology modules have finite length. Then* $\dim(R) \leq n$.

Theorem. *Let R be a regular ring that contains a field. Then R is a direct summand of every module-finite extension algebra.*

The second result is easy in characteristic 0 by other means, where it holds for normal rings. The characteristic p result was not established until the early 1973, however. The first argument is in [M. Hochster, *Contracted ideals from integral extensions of regular rings*, Nagoya Math. J. **51** (1973) pp. 25–43]. The mixed characteristic case of this problem remains an open question in dimension ≥ 4.