Math 711: Lecture of December 16, 2004

The new intersection theorem is known even for rings of mixed characteristic, but the proof in the mixed characteristic case, which is due to Paul Roberts, is quite difficult. Cf. [P. Roberts, *Le théorème d'intersection*, C. R. Acad. Sc. Paris Sér. I **304** (1987) pp. 177–180], [______, *Intersection theorems,* in *Commutative Algebra*, Math. Sci. Research Inst. Publ. **15**, Springer-Verlag, New York-Berlin-Heidelberg (1989) pp. 417–436], and [______, *Multiplicities and Chern classes in local algebra*, Cambridge Tracts in Mathematics **133**, Cambridge University Press, Cambridge, England, 1998]. Here we give the argument in equal characteristic. It is valid for any local ring that has a big Cohen-Macaulay module.

Proof of the new intersection theorem. Let

$$0 \to G_n \to \cdots \to G_1 \to G_0 \to 0$$

be the given free complex. Recall that $H_0(G_\bullet) \neq 0$ and choose a minimal generator u of $H_0(G_\bullet)$. Of course, u is killed by some power of the maximal ideal m of R. (In fact, the proof does not use that $H_0(G_\bullet)$ has finite length: only that it is nonzero and has a minimal generator u that is killed by a power of m.) Some element of G_0 maps onto u, and it must be a minimal generator of G_0. Thus, we may write $G_0 = Re_1 \oplus G_0'$ where both summands are R-free. We can choose an integer N_0 so large that $m^{N_0}u = 0$ and $m^{N_0}H_i(G_\bullet) = 0$ for $i \geq 1$. Let Z_i and B_i denote the modules of cycles and boundaries respectively in G_i, so that $H_i(G_\bullet) = Z_i/B_i$. Then $m^{N_0}e_1 \in B_0$ and $m^{N_0}Z_i \subseteq B_i$ for $i \geq 1$.

For each $i \geq 1$, we may use the Artin-Rees Lemma to choose N so large that

$$m^N G_i \cap Z_i \subseteq m^{N_0} Z_i.$$

Since we need only be concerned with $i \leq n$, we may choose $N \geq N_0$ so that for all i, $1 \leq i \leq n$, $m^N G_i \cap Z_i \subseteq m^{N_0} Z_i \subseteq B_i$.

Next, we choose a system of parameters $\underline{x} = x_1, \ldots, x_d$ for R inside m^N. We can do this by first choosing any system of parameters and then replacing every element by its Nth power. Recall that what we want to prove is that $d \leq n$. Assume, to the contrary, that $d > n$. We shall obtain a contradiction. Let $\mathcal{K}_\bullet(\underline{x}; R)$ denote the Koszul complex on $x_1, \ldots, x_d$. First note that we have a commutative diagram

$$\begin{array}{ccc} G_0 & \longrightarrow & H_0(G_\bullet) \to 0 \\ \uparrow & & \uparrow \\ R & \longrightarrow & R/(\underline{x}) \to 0 \end{array}$$

where the vertical arrow on the left takes $1 \in R$ to $e_1 \in G_0$ and the vertical arrow on the right maps $\overline{1} \in R/(\underline{x})$ to u (the second map exists since u is killed by m^{N_0} and $(\underline{x}) \subseteq m^N \subseteq m^{N_0}$).

Think of the copy of R on the left in the bottom row as $\mathcal{K}_0(\underline{x};\, R)$. We shall show by induction that this diagram extends to a map from the Koszul complex $\mathcal{K}(\underline{x};\, R)$ to $G_\bullet$:

$$\begin{array}{ccccccccc}
0 & \longrightarrow \cdots \longrightarrow & G_n & \longrightarrow \cdots \longrightarrow & G_0 & \longrightarrow & H_0(G_\bullet) & \longrightarrow & 0 \\
\uparrow & & \uparrow & & \uparrow & & \uparrow & & \\
\mathcal{K}_d(\underline{x};\, R) & \longrightarrow \cdots \longrightarrow & \mathcal{K}_n(\underline{x};\, R) & \longrightarrow \cdots \longrightarrow & R & \longrightarrow & R/(\underline{x}) & \longrightarrow & 0
\end{array}$$

Assume that the vertical maps have been constructed, starting on the right, up to and including $\phi : \mathcal{K}_i(\underline{x};\, R) \to G_i$ so that the diagram commutes. Thus, we have:

$$\begin{array}{ccccccccc}
\longrightarrow & G_{i+1} & \longrightarrow & G_i & \longrightarrow & G_{i-1} & \longrightarrow \\
& {\scriptstyle ?}\big\uparrow & & {\scriptstyle \phi_i}\big\uparrow & & {\scriptstyle \phi_{i-1}}\big\uparrow & \\
\longrightarrow & \mathcal{K}_{i+1}(\underline{x};\, R) & \longrightarrow & \mathcal{K}_i(\underline{x};\, R) & \longrightarrow & \mathcal{K}_{i-1}(\underline{x};\, R) & \longrightarrow
\end{array}$$

We want to define the map ϕ_{i+1} so that the square on the left will commute. Consider a free generator f of $\mathcal{K}_{i+1}(\underline{x};\, R)$. The matrices of the maps in the Koszul complex have entries each of which is 0 or $\pm x_j$ for some j: thus all entries are in $(x_1,\, \ldots\, , x_d) \subseteq m^N$. Hence, the image of f in $\mathcal{K}_i(\underline{x};\, R)$ is actually in $(x_1,\, \ldots\, , x_d)\mathcal{K}_i(\underline{x};\, R)$, and so the image of f in G_i is in $m^N G_i$. It is also in Z_i, that is, the image of f in G_{i-1} is 0, because we view the map $\mathcal{K}_{i+1}(\underline{x};\, R) \to G_{i-1}$ as factoring

$$\mathcal{K}_{i+1}(\underline{x};\, R) \to \mathcal{K}_i(\underline{x};\, R) \to \mathcal{K}_{i-1}(\underline{x};\, R) \to G_{i-1}$$

and the composition of the two maps on the left is 0. Thus, the image of f in G_i is in $m^N G_i \cap Z_i \subseteq B_i$, by our choice of N, and so we can choose $f' \in G_{i+1}$ such that f' maps to the image of f in G_i. We define the value of ϕ_{i+1} on f to be f'. We repeat this argument for every element in a free basis for $\mathcal{K}(\underline{x};\, R)$, and so define ϕ_{i+1}.

We now have the desired map of complexes. Now suppose that $\mathcal{B}$ is a big Cohen-Macaulay module for R (all we need is that $x_1,\, \ldots\, , x_d$ is a regular sequence on $\mathcal{B}$ and that some element $v \in \mathcal{B} - (x_1,\, \ldots\, , x_d)\mathcal{B}$). The image of v in $\mathcal{B}/(x_1,\, \ldots\, , x_d)\mathcal{B}$ spans a cyclic module of finite length: we replace v by a multiple, which we still denote by v, which is in the socle. That is, without loss of generality, we may assume that $v \notin (x_1,\, \ldots\, , x_d)B$ but $mv \subseteq (x_1,\, \ldots\, , x_d)B$.

Next note that because u is a minimal generator of $H = H_0(G_\bullet)$, it has nonzero image in the K-vector space H/mH. Hence, there is a map of H/mH onto $K\overline{v} \subseteq B/(x_1,\, \ldots\, , x_d)\mathcal{B}$ that sends the image of u to v, and so there is a map $H \to \mathcal{B}/(x_1,\, \ldots\, , x_d)$ that sends $u \to v$. This will lift to a map of $G_0 = Re_1 \oplus G_0'$ to B sending e_1 to v. We now extend this map to a map of complexes $G_\bullet \to \mathcal{K}_\bullet(\underline{x};\, \mathcal{B})$, which we can do because $G_\bullet$ is free and $\mathcal{K}(\underline{x};\, \mathcal{B})$ is acyclic. We can then compose with the map of complexes $\mathcal{K}_\bullet(\underline{x};\, R) \to G_\bullet$ already constructed to get a map of complexes $\mathcal{K}_\bullet(\underline{x};\, R) \to \mathcal{K}_\bullet(\underline{x};\, \mathcal{B})$ that sends $1 \in R = \mathcal{K}_0(\underline{x};\, R)$ to $v \in \mathcal{B} = \mathcal{K}_0(\underline{x};\, \mathcal{B})$. Note, however, that because of the assumption that $d > n$, the last map factors through $G_d = 0$, and so is the zero map.

We can give another map of complexes $\mathcal{K}_\bullet(\underline{x};\, R) \to \mathcal{K}_\bullet(\underline{x};\, \mathcal{B})$ that sends $1 \in R = \mathcal{K}_0(\underline{x};\, R)$ to $v \in \mathcal{B} = \mathcal{K}_0(\underline{x};\, \mathcal{B})$ as follows: take the map $R \to \mathcal{B}$ that sends 1 to v and tensor over R with the complex $\mathcal{K}_\bullet(\underline{x};\, R)$ to get a map of complexes. But now the last map sends $1 \in R = \mathcal{K}_d(\underline{x};\, R)$ to $v \in \mathcal{B} = \mathcal{K}_d(\underline{x};\, \mathcal{B})$, and we are close to the desired contradiction. Because the Koszul complex of R is free and the Koszul complex of $\mathcal{B}$ is acyclic, these two maps of complexes are homotopic. Therefore, the difference of the two maps $\mathcal{K}_d(\underline{x};\, R) \to \mathcal{K}_d(\underline{x};\, \mathcal{B})$ is the composition of a map $h : \mathcal{K}_{d-1}(\underline{x};\, R) \to \mathcal{K}_d(\underline{x};\, \mathcal{B})$ with the map $\delta : \mathcal{K}_d(\underline{x};\, R) \to \mathcal{K}_{d-1}(\underline{x};\, R)$ from the complex, where h is one of the maps giving the homotopy. (There is another relevant map involved in the homotopy, from $\mathcal{K}_d(\underline{x};\, R) \to \mathcal{K}_{d+1}(\underline{x};\, \mathcal{B})$, but it is the zero map, since $\mathcal{K}_{d+1}(\underline{x};\, ,\, \mathcal{B}) = 0$.) Since the image of δ is contained in $(x_1, \ldots, x_d)\mathcal{K}_{d-1}(\underline{x};\, R)$, the image of $h \circ \delta$ is contained in $(x_1, \ldots, x_d)\mathcal{B}$, which shows that $v \in (x_1, \ldots, x_d)\mathcal{B}$, a contradiction. $\square$

From this one can easily deduce the original intersection theorem of Peskine-Szpiro, which was proved in their joint thesis. (From Roberts's work this is known without the equicharacteristic restriction.)

Theorem (Peskine-Szpiro intersection theorem). *Let R be an equicharacteristic local ring and let $M \neq 0$, $N \neq 0$ be finitely generated R-modules such that $\mathrm{pd}_R M$ is finite and $M \otimes_R N$ has finite length. Then* $\dim(N) \leq \mathrm{pd}_R M$.

This follows at once from the new intersection theorem: let $I = \mathrm{Ann}_R N$. Then R/I has the same dimension as N, and the length of $M \otimes_R (R/I)$ is finite, since the length of $M \otimes_R N$ is finite if and only if the sum of their annihilators is primary to the maximal ideal. Therefore we may replace N by R/I. Take a minimal free resolution of M, which has length $n = \mathrm{pd}_R M$, and tensor with $S = R/I$. One gets a free complex $G_\bullet$ over S which has finite length homology: the homology is $\mathrm{Tor}^R_\bullet(M,\, N)$ which is killed by $\mathrm{Ann}_R(M) + \mathrm{Ann}_R(N)$. Thus, the new intersection theorem may be applied over S to conclude that $\dim(R/I) \leq n$.

Peskine and Szpiro show that the intersection theorem implies M. Auslander's zerodivisor conjecture: that conjecture asserts that if R is local, and $M \neq 0$ is finitely generated and has finite projective dimension, then any zerodivisor in the ring R is a zerodivisor on M. It follows that a regular sequence on M must be a regular sequence in R.

They also proved that the intersection theorem implies an affirmative answer to a question of Bass (eventually known as Bass's conjecture): if a local ring possesses a nonzero module of finite injective dimension then the ring must be Cohen-Macaulay. (The converse was known: a Cohen-Macaulay local ring does possess a finitely generated module of finite injective dimension.)

It is worth noting that the new intersection theorem implies the Krull height theorem in a very simple way. Let I be an n generator ideal of a Noetherian ring R. We want to see that every minimal prime of I has height at most n. We may localize the minimal prime in question. The result therefore asserts that if $x_1, \ldots, x_n$ generates an ideal primary to the maximal ideal m of the local ring (R, m), then $\dim(R) \leq n$. This follows from considering the Koszul complex $\mathcal{K}(x_1, \ldots, x_n;\, R)$: it has finite length homology, since the homology is killed by the m-primary ideal $(x_1, \ldots, x_n)R$, and so $\dim(R) \leq n$, as required.

We remarked during the course of the proof the new intersection theorem that the argument shows more. The difference may seem rather technical, but it turns out to be important. The stronger result is:

Theorem (improved new intersection theorem). *Let R be a local ring that contains a field. Let $0 \to G_n \to \cdots \to G_0 \to 0$ be a finite complex of finitely generated free modules such that $H_0(G_0) \neq 0$ and has a minimal generator that is killed by a power of the maximal ideal, and such that all the higher homology modules have finite length. Then* $\dim(R) \leq n$.

This result is *not* known in mixed characteristic. It can be used to prove the Evans-Griffith Syzygy Theorem, which asserts that a k th module of syzygies of a finitely generated module over a regular local ring, if it is not free, must have rank at least k. The Evans-Griffith Syzygy theorem remains open in mixed characteristic. Cf. [M. Hochster, *Canonical elements in local cohomology modules and the direct summand conjecture*, J. of Algebra **84** (1983) pp. 503–553] for the argument deducing the syzygy theorem from the improved new intersection theorem, and [E. G. Evans and P. Griffith, *Syzygies*, London Math. Soc. Lecture Note Series **106**, Cambridge Univ. Press, Cambridge, England, 1985] for further background and variant results.

We next want to turn our attention to the question, is every regular local ring a direct summand of every module-finite extension ring?

We begin by noting several reductions that are possible in considering this problem. First, $R \hookrightarrow S$ splits if and only the map

$$\mathrm{Hom}_R(S,\, R) \to \mathrm{Hom}_R(R,\, R)$$

is onto. (The map $g \in \mathrm{Hom}_R(S,\, R)$ that maps to $\mathbf{1}_R$ is the splitting of $R \hookrightarrow S$.) Since localization commutes with Hom for finitely generated modules over a Noetherian ring, we may reduce to the case where R is local. Likewise, we may reduce to the complete case, since the completion of R is faithfully flat over R and flat base change commutes with Hom for finitely generated modules over a Noetherian ring.

Thus, we may assume that the regular ring is local or even complete local. Next, we may assume that S is a domain. For we may kill a minimal prime ideal P of S disjoint from $R - \{0\}$, so that we have $R \hookrightarrow S \twoheadrightarrow S/P$ and we still have $R \hookrightarrow S/P$. The composition of a splitting $g : S/P \twoheadrightarrow R$ with the map $S \twoheadrightarrow S/P$ will split $R \hookrightarrow S$.

We shall next use the existence of big Cohen-Macaulay modules to prove that regular rings are direct summands of their module-finite extensions in equal characteristic. But we first want to note that this proof is mainly of interest in characteristic $p > 0$. In equal characteristic 0, even normal rings are direct summands of their module-finite extensions, by a very simple argument: if R is a normal domain containing the rational numbers, S is a module-finite extension domain, and the fraction fields of R and S are $\mathcal{K}$ and $\mathcal{L}$, respectively, then if $d = [\mathcal{L} : \mathcal{K}]$, the map $\dfrac{1}{d}\mathrm{Trace}_{\mathcal{L}/\mathcal{K}}$, field trace from $\mathcal{L}$ to $\mathcal{K}$, is an R-linear retraction from S to R. (The trace of $\lambda \in L$ is the trace of multiplication by λ as a $\mathcal{K}$-linear transformation on the d-dimensional $\mathcal{K}$-vector space $\mathcal{L}$. Thus, the trace of 1 is d.) The map is *a priori* defined from $\mathcal{L} \to \mathcal{K}$ and is $\mathcal{K}$-linear. We need to check that if $s \in S$ then the trace of s is in R, not just in $\mathcal{K}$.

One argument is as follows: let f be the minimal polynomial of s over $\mathcal{K}$. Since R is normal, the coefficients are in R. Let $\mathcal{L}_0 = \mathcal{K}[s]$. Then $\mathrm{Trace}_{\mathcal{L}/\mathcal{K}}(s) = [\mathcal{L} : \mathcal{L}_0]\mathrm{Trace}_{\mathcal{L}_0/\mathcal{K}}(s)$, and the latter is the sum of the roots of f. Since the roots of f are all integral over R, $\mathrm{Trace}_{\mathcal{L}_0/\mathcal{K}}(s)$ is integral over R as well as being in $\mathcal{K}$, and so is in R. $\square$

An alternative argument in the Noetherian case uses that a normal ring R is an intersection of discrete valuation rings $V \subseteq \mathcal{K}$: one may let V run through the localizations of R at its height one primes. One needs to see that $\mathrm{Trace}_{\mathcal{L}/\mathcal{K}}(s)$ is in each such V. Note that $V \subseteq \mathcal{K} \subseteq \mathcal{L}$ and $S \subseteq \mathcal{L}$, and so we may form the ring $W = V[S] \subseteq \mathcal{L}$ generated over V by the elements of S. In fact, a finite set of generators for S as an R-module will also be a finite set of generators for W as a V-module. We may work with V and W instead of R and S. But now W is free over V, since W is finitely generated and torsion-free over V and V is a principal ideal domain, and so the field trace of multiplication by s may be calculated using a free basis for W over V as the basis for $\mathcal{L}$ over $\mathcal{K}$. The entries of the matrix of multiplication by s will all be in V, and so the field trace of s is in V, as required. $\square$