Math 711: Lecture of December 17, 2004

Let (R, m, K) be a regular local ring of dimension d and let $x_1, \ldots, x_d$ be a regular system of parameters, so that $(x_1, \ldots, x_d) = m$. We want to give a very down-to-earth criterion for when the regular ring R splits from a module-finite extension S. Before doing so, we establish some properties of regular sequences on a module. For some of these, we refer to the Lecture Notes, Problem Sets and Problem Set Solutions from Math 615.

Proposition. *Let R be a ring, let M be an R-module, and let $x_1, \ldots, x_d$ be a sequence of elements of R. Suppose that $x_1, \ldots, x_d$ is a regular sequence on M.*

(a) *For all positive integers $a_1, \ldots, a_d$, $x_1^{a_1}, \ldots, x_d^{a_d}$ is a regular sequence on M, and if $\sum_{j=1}^{d} x_j^{a_j} u_j = 0$, where the $u_j \in M$, then for all j, $1 \le j \le d$,*

$$u_j \in (x_1^{a_1}, \ldots, x_{j-1}^{a_{j-1}}, x_{j+1}^{a_{j+1}}, \ldots, x_d^{a_d})M.$$

(b) *Let $a_1, \ldots, a_d$ be nonnegative integers and let $b_1, \ldots, b_d$ be positive integers such that $b_j > a_j$, $1 \le j \le d$. Then*

$$(x_1^{b_1}, \ldots, x_d^{b_d})M :_M x_1^{a_1} \cdots x_d^{a_d} = (x_1^{b_1-a_1}, \ldots, x_d^{b_d-a_d})M.$$

(c) *For every integer $t \ge 1$, the map*

$$M/(x_1^{t-1}, \ldots, x_d^{t-1})M \to M/(x_1^{t}, \ldots, x_d^{t})M$$

induced by multiplication by $x_1 \cdots x_d$ is injective.

(d) *Let $y = x_1 \cdots x_d$. For every positive integer t,*

$$(x_1^{t}, \ldots, x_d^{t})M :_M (x_1, \ldots, x_d) = (x_1^{t}, \ldots, x_d^{t}, y^{t-1})M.$$

(e) *The relations on $x_1, \ldots, x_d$ in $R_t = R/I_t$, where $I_t = (x_1^t, \ldots . x_d^t)$, are spanned by the Koszul relations $x_j e_i - x_i e_j$ and the relations $x_i^{t-1} e_i$, where $e_1, \ldots, e_d$ is the standard free basis for R_t^d.*

(f) *Let $J = (x_1^{b_1}, \ldots, x_d^{b_d})$ for positive integers $b_1, \ldots, b_d$, suppose that $1 \le h \le d$, and let $a_1, \ldots, a_h$ be nonnegative integers such that $a_i < b_i$, $1 \le i \le h$. Then*

$$\bigcap_{i=1}^{h} \big((x_i^{a_i}) + J\big) = (x_1^{a_1} \cdots x_h^{a_h}) + J.$$

In particular, if $h < d$, $b_{h+1} = t-1$ while the other b_i are all t, so that $J = (x_{h+1}^{t-1}) + I_t$, and $a_1 = \cdots = a_h = t-1$, we obtain that for $1 \le h < d$,

$$\bigcap_{i=1}^{h} \big((x_i^{t-1}) + (x_{h+1}^{t-1}) + I_t\big) = (x_1^{t-1} \cdots x_h^{t-1}) + (x_{h+1}^{t-1}) + I_t.$$

(g) *With the same hypothesis and notation as in part (e), every R_t-linear homomorphism of $(x_1, \ldots, x_d)R_t$ into R_t is induced by multiplication by some $r \in R_t$. That is, if we start with the inclusion $(x_1, \ldots, x_d)R_t \hookrightarrow R_t$ then the map induced by applying $\mathrm{Hom}_{R_t}(\_\,, R_t)$ is surjective. This is equivalent to the assertion that*

$$\mathrm{Ext}^1_{R_t}\big(R/(x_1, \ldots, x_d), R_t\big) = 0.$$

Proof. The first statement in part (a) is the Extra Credit problem in Problem Set #3 from Math 615, and, given the first statement, the second statement is immediately reduced to the case where all the $a_i = 1$. We use induction on d. If $d = 1$ and $u_1 x_1 = 0$, then $u_1 = 0$. Suppose that

$$x_1 u_1 + \cdots + x_d u_d = 0.$$

If $j = d$ the result is immediate from the definition of a regular sequence. If not, we write

$$u_d = x_1 v_1 + \cdots + x_{d-1} v_{d-1},$$

and the original relation becomes

$$x_1(u_1 + x_d v_1) + \cdots + x_{d-1}(u_{d-1} + x_d v_{d-1}) = 0.$$

The result now follows from the induction hypothesis.

We prove part (b) using induction on the number of elements among $a_1, \ldots, a_d$ that are not zero. If all of the elements are zero the statement is obvious. Now suppose that $a_j > 0$. Suppose that

$$x_1^{a_1} \cdots x_d^{a_d} u = \sum_{i=1}^{d} x_i^{b_i} u_i$$

with $u, u_1, \ldots, u_d \in M$. Let $x = \prod_{k \neq j} x_k^{a_k}$. Then

$$x_j^{a_j}(xu - x_j^{b_j - a_j} u_j) = \sum_{i \neq j} x_i^{b_i} u_i.$$

From the second statement in part (a), we see that

$$xu - x_j^{b_j - a_j} u_j = \sum_{i \neq j} x_i^{b_i} v_i,$$

for suitable $v_i \in M$, i.e.,

$$xu = x_j^{b_j - a_j} u_j + \sum_{i \neq j} x_i^{b_i} v_i.$$

This is the same type of equality that we started with, except that the exponent on x_j on the right has been reduced to $b_j - a_j$, and one more exponent occurring in x on the left, namely, the exponent on x_j, is now zero. The desired result is now immediate from the induction hypothesis.

Part (c) is a special case of part (b): the case where all the b_j are t and all the a_j are 1.

To prove (d), let $J = (x_1^t, \dots, x_d^t)$, let $I_h = (x_{d-h+1}, \dots, x_d)$, $0 \le h \le d$, where I_0 is interpreted to be (0). Let $y_h = x_{d-h+1} \cdots x_d$, $0 \le h \le d$, where y_0 is interpreted to be 1 and $y_d = y$. We shall prove by induction on h that $JM :_M I_h = (J, y_h^{t-1})M$. The case $j = 0$ is clear: the left hand side is $JM :_M 0 = M$, and the right hand side is $(J, R)M = M$. For the inductive step, suppose that we know the result for a certain $h < d$ and that $I_{h+1}u \subseteq JM$. Then $I_h u \subseteq JM$, and so we can write u in the form $v + y_h^{t-1}w$, where $v \in JM$ and $w \in M$. To complete the argument is suffices to show that $y_h^{t-1}w \in y_{h+1}^t M + JM$. But

$$x_{d-h}(v + y_h^{t-1}w) \in JM,$$

and since $v \in JM$ we have that

$$x_{d-h}y_h^{t-1}w \in JM.$$

By part (b),

$$w \in (x_1^t, \dots, x_{d-h-1}^t, x_{d-h}^{t-1}, x_{d-h+1}, \dots, x_d)M,$$

Since the first $d-h-1$ generators are already in J, the product of y_h^{t-1} with x_{d-h}^{t-1} is y_{h+1}^{t-1}, and the product of y_h^{t-1} with each of the remaining generators is in J, the result is proved.

For part (e), note that a relation on $x_1, \dots, x_d$ mod $(x_1^t, \dots, x_d^t)R$ is expressed by an equation of the form

$$\sum_j a_j x_j = \sum_j b_j x_j^t$$

where the a_j, $b_j \in R$, and this can be rewritten as

$$\sum_j (a_j - b_j x_j^{t-1})x_j = 0,$$

where in sums indexed by j, j runs from 1 to d. This shows that $\sum_j (a_j - b_j x_j^{t-1})e_j$ is a linear combination of Koszul relations on $x_1, \dots, x_d$ even over R, and subtracting from the original relation on the x_j evidently produces a relation which is a linear combination of the relations $x_j^{t-1}e_j$.

For part (f) we use induction on h. If $h = 1$ the result is tautological. At the inductive step what we need to show is that $\big((\mu) + J\big) \cap \big((\nu) + J\big) = (\mu\nu) + J$ where $\mu = x_1^{a_1} \cdots x_h^{a_h}$ and $\nu = x_{h+1}^{a_{h+1}}$. We need only show $\subseteq$, since the opposite inclusion is obvious. For an element u of the intersection we have $u = \mu v + j = \nu w + j'$ where $j, j' \in J$ and then $\mu v \in (\nu) + J$ and so $v \in \big((\nu) + J\big) : R\mu$. From part (b), this is

$$(x_1^{b_1 - a_1}, \dots, x_h^{b_h - a_h}, x_{h+1}^{a_{h+1}}) + J.$$

But μ times any of the first h generators is in J, and so $\mu v \in (\mu\nu) + J$, and the same holds for $u = \mu v + j$.

For part (g) suppose that $r_1, \dots, r_d$ are elements of R such that there is a homomorphism $(x_1, \dots, x_d)R_t \to R_t$ whose values on the images of $x_1, \dots, x_d$ are represented by the images of $r_1, \dots, r_d$. Then for all i and j,

$$r_j x_i - r_i x_j \in (x_1^t, \dots, x_d^t),$$

and for all j,

$$r_j x_j^{t-1} \in (x_1^t, \dots, x_d^t).$$

From part (b), it follows that

$$r_j \in (x_1^t, \dots, x_{j-1}^t, x_j, x_{j+1}^t, \dots, x_d^t),$$

and since we are free to alter each r_j by subtracting an element of I_t, we may assume without loss of generality that $r_j = s_j x_j$ for all j. We then find that $s_j x_j x_i - s_i x_i x_j = (s_j - s_i) x_i x_j \in I_t$ for all i, j, and so, again by part (b),

$$s_j - s_i \in (x_i^{t-1}, x_j^{t-1}) + (x_k^t : k \neq i, j).$$

We shall show by induction on h, $1 \le h \le d$, that $s_1, \dots, s_h$ can be replaced by a single element $s \in S$ such that, mod I_t, $sx_i = s_i x_i$, $1 \le i \le h$. This is clear if $h = 1$. Now suppose that we have proved the result for $1 \le h < d$. Then we may assume that $s_1 = \cdots = s_h = s$. Then $s_{h+1} - s \in (x_i^{t-1}, x_{h+1}^{t-1}) + (x_k^t : k \neq i, h+1)$ for $1 \le i \le h$, and we may intersect all of these ideals. The intersection is $\bigl((x_1 \cdots x_h)^{t-1}\bigr) + (x_{h+1}^{t-1}) + I_t$ by part (f). Thus, we may write

$$s_{h+1} - s = (x_1 \cdots x_h)^{t-1} v + x_{h+1}^{t-1} w + z$$

where $z \in I_t$, and so we may let

$$s' = s_{h+1} - x_{h+1}^{t-1} w - z = s + (x_1 \cdots x_h)^{t-1} v.$$

We now see that we can use s' instead of s or s_{h+1}, since $s' x_i \equiv s x_i$ mod I_t for $i \le h$ (we have that x_i^{t-1} divides $s' - s$), and $s' x_{h+1} \equiv s_{h+1} x_{h+1}$ mod I_t (we have that, mod I_t, x_{h+1}^{t-1} divides $s_{h+1} - s$). This completes the inductive step. The homomorphism coincides with multiplication by s, where s is the element that is obtained in the case where $h = d$. $\square$

We give an alternative proof of part (g) of the preceding Proposition due to Yongwei Yao. This result is more general, but uses machinery other than elementary properties of regular sequences. The result of part (g) follows from the last statement in part (b) of the Proposition below in the case where $y_j = x_j^t$, $1 \le j \le d$, $M = R/(x_1, \dots, x_d)R$ and $N = R$.

Proposition. *Let R be a ring, and let M and N be R-modules.*

(a) *Let $y \in R$ be such that $yM = 0$ and y is a nonzerodivisor on both R and N. Then $\mathrm{Ext}^i_{R/yR}(M,\, N/yN) \cong \mathrm{Ext}^{i+1}_R(M,\, N)$ for all $i \geq 0$.*

(b) *Let $\underline{y} = y_1, \, \ldots \, , y_d \in R$ be a regular sequence on R and N such that $(y_1, \, \ldots \, , y_d)M = 0$. Then $\mathrm{Ext}^i_{R/(\underline{y})}\bigl(M,\, N/(\underline{y})N\bigr) \cong \mathrm{Ext}^{i+d}_R(M,\, N)$. In particular, if $\mathrm{pd}_R M \leq d$, then for all $j > 0$, $\mathrm{Ext}^j_{R/(\underline{y})}\bigl(M,\, N/(\underline{y})N\bigr) = 0$.*

Proof. The first statement in part (b) is immediate from part (a) by induction on d, and the second statement in part (b) is immediate from the first statement. It remains only to prove part (a).

We give a separate proof when $i = 0$, although the result in this case can also be deduced along the same lines as in the argument below for the case $i > 0$.

If $i = 0$, note that applying $\mathrm{Hom}_R(M, \, \_)$ to the short exact sequence

$$0 \to N \xrightarrow{y} N \to N/yN \to 0$$

yields a long exact sequence part of whose beginning is

$$\mathrm{Hom}_R(M,\, N) \to \mathrm{Hom}_R(M,\, N/yN) \xrightarrow{\delta} \mathrm{Ext}^1_R(M,\, N) \xrightarrow{y} \mathrm{Ext}^1_R(M,\, N) \to \cdots .$$

Note that $\mathrm{Hom}_R(M,\, N) = 0$, that $\mathrm{Hom}_R(M,\, N/yN) \cong \mathrm{Hom}_{R/yR}(M,\, N/yN)$, and that multiplication by y is the zero map on $\mathrm{Ext}^1_R(M,\, N)$, so that δ induces an isomorphism $\mathrm{Hom}_R(M,\, N/yN) \xrightarrow{\cong} \mathrm{Ext}^1_R(M,\, N)$ as required. Henceforth we assume that $i \geq 1$.

Let $0 \to N \to E^0 \to E^1 \to \cdots \to E^n \to \cdots$ be an injective resolution of N, and let $E^\bullet$ be the complex $0 \to E^0 \to E^1 \to \cdots \to E^n \to \cdots$. The cohomology of the complex $\mathrm{Hom}_R(R/yR,\, E^\bullet)$ is $\mathrm{Ext}^\bullet_R(R/yR,\, N)$, which we may compute from the projective resolution $0 \to R \xrightarrow{y} R \to 0$ of R/yR: thus, $\mathrm{Hom}(R/yR,\, N) = 0$ and $\mathrm{Ext}^1_R(R/yR,\, N) \cong N/yN$, while $\mathrm{Ext}^i_R(R/yR,\, N) = 0$ for $i \geq 2$. Let $\overline{E}_i$ denote

$$\mathrm{Hom}_R(R/yR,\, E_i) \cong \mathrm{Ann}_{E_i} y,$$

Note that if E is injective over R, then $\overline{E} = \mathrm{Hom}_R(R/yR,\, E)$ is an injective module over R/yR: in fact, the functor $Q \mapsto \mathrm{Hom}_{R/yR}(Q,\, \overline{E})$ on (R/yR)-modules is isomorphic with the functor $Q \mapsto \mathrm{Hom}_R(Q, E)$ on (R/yR)-modules, since the image of a map $Q \to E$ must consist entirely of elements killed by y and so must be contained in $\overline{E}$.

Since $H^0(\overline{E}^\bullet) = 0$, $\overline{E}^0$ injects into $\overline{E}^1$ as a submodule of the module of cocycles $Z^1 \subseteq \overline{E}^1$: the image of $\overline{E}^0$ is the module of coboundaries B^1. Since $\overline{E}_0$ is injective over R/yR, it splits from $\overline{E}^1$, and so $\overline{E}^{1\prime} = \overline{E}^1/B^1$ is injective. Now, $Z^1/B^1 = Z^1/\overline{E}^1 = \mathrm{Ext}^1_R(R/yR,\, N) \cong N/yN$, and is the kernel of the map from $\overline{E}^{1\prime} \to \overline{E}^2$. This yields an exact sequence $0 \to N/yN \to \overline{E}^{1\prime} \to \overline{E}^2 \to \overline{E}^3 \cdots$ which is an injective resolution of N/yN over R/yR, with the numbering shifted by one from the usual numbering.

If we apply $\operatorname{Hom}_{R/yR}(M,\ \_)$ to

$$0 \to \overline{E}^{1\prime} \to \overline{E}^{2} \to \overline{E}^{3} \to \cdots$$

and take cohomology, we get the modules $\operatorname{Ext}^{\bullet}_{R/yR}(M,\ N/yN)$, with the term of the complex indexed by $i+1$ corresponding to $\operatorname{Ext}^{i}_{R/yR}(M,\ N/yN)$. We get the same cohomology at the spots indexed by 2, 3, ... by applying $\operatorname{Hom}_R(M,\ \_)$ to

$$0 \to E^0 \to E^1 \to E^2 \to E^3 \cdots \to .$$

Again, because M is killed by y, a map of M to E_i is the same as a map of M to $\overline{E}_i$. This cohomology will be $\operatorname{Ext}^{i+1}_R(M,\ N)$ as claimed, $i \geq 1$. When $i = 1$, one needs to make the observation as well that the image of $\operatorname{Hom}_R(M,\ \overline{E}_1) \to \operatorname{Hom}_R(M,\ \overline{E}_2)$ is the same as the image of $\operatorname{Hom}_R(M,\ \overline{E}_1{}') \to \operatorname{Hom}_R(M,\ \overline{E}_2)$, since $\overline{E}_0$ is in $\operatorname{Ker}(\overline{E}^1 \to \overline{E}^2)$ and $0 \to \overline{E}^0 \to \overline{E}^1 \to \overline{E}_1{}' \to 0$ is split exact. $\square$

Theorem. *Let let $(R,\ m,\ K)$ be a regular ring and let $x_1, \ldots, x_d$ be a regular system of parameters. Let $I_t = (x_1^t, \ldots, x_d^t)R$ and let $R_t = R/I_t$. Let $y = x_1 \cdots x_d$.*

(a) *The ring R_t has a one-dimensional socle represented by the element y^{t-1}. Hence, every ideal of R strictly larger than I_t contains y^{t-1}.*

(b) *The ring R_t is injective as an R_t-module.*

(c) *The map of R-modules $R_t \to R_{t+1}$ induced by multiplication by y mapping $R \to R$ is injective, and the direct limit $E = \varinjlim_t R_t$, where every map is induced by multiplication by y, is an injective R-module.*

(d) *If R is complete and E is as in part (c), the map $R \to \operatorname{Hom}_R(E,\ E)$ is an isomorphism.*

Proof. Since R is regular, it is Cohen-Macaulay, and every system of parameters is a regular sequence.

For (a), since $m = (x_1, \ldots, x_d)$, the result follows from the fact that

$$I_t :_R (x_1, \ldots, x_d) = I_t + y^{t-1}R,$$

which is immediate from part (d) of the first Proposition above.

To prove (b), first note that by part (2) of the third Proposition on the fifth page of the Lecture Notes of March 22 from Math 615, to establish that R_t is injective it suffices to show that $\operatorname{Ext}^1_{R_t}(R_t/\mathfrak{A},\ R_t) = 0$ for every ideal $\mathfrak{A}$ of R_t. Now if M has a finite filtration with factors M_j such that $\operatorname{Ext}^t_{R_t}(M_j,\ N) = 0$ for all j, then $\operatorname{Ext}^t_{R_t}(M,\ N) = 0$ (if there are just two factors this follows from the long exact sequence for Ext; the general case follows by induction on the number of factors). Since R_t is Artinian, $R_t/\mathfrak{A}$ has a finite filtration in which all the factors are copies of K. It follows that R_t is injective if $\operatorname{Ext}^1_{R_t}(K,\ R_t) = 0$. But this is part (g) of the first Proposition.

The first statement in part (c) follows from part (d) of the Proposition on the first page. To see that E is injective, note that by the Proposition on the second page of the Lecture Notes of March 19 from Math 615, it suffices to show that every map of an ideal I of R to E extends to R. Since I is finitely generated, the map $I \to E$ factors $I \to R_t \hookrightarrow E$ for all sufficiently large t. The map $I \to R_t$ kills I_tI. For $s \gg 0$, $I_s \cap I \subseteq I_tI$ by the Artin-Rees lemma, since I_s is contained in arbitrarily high powers of m for $s \gg 0$. Thus, we have an induced map $I/(I_s \cap I) \twoheadrightarrow I/I_tI \to R_t \hookrightarrow R_s$ for $s \gg 0$. Here, $I/(I_s \cap I) \cong (I + I_s)/I_s \subseteq R/I_s$. Since R_s is injective as an R_s-module, this map extends to a map $R/I_s \to R_s \hookrightarrow E$, giving a map $R \twoheadrightarrow R/I_s \to R_s \hookrightarrow E$. This map extends the original map $I \to E_t \subseteq E$.

It remains to prove part (d). First note that $\mathrm{Ann}_E I_t$ is the copy of R_t in the direct limit system. It contains the copy of R_t, obviously. The fact that it agrees with R_t is equivalent to the assertion that for all $s > t$, the annihilator of I_t in R/I_s is spanned by the image of y^{s-t} (since this element spans the image of R_t in R_s in the direct limit system), i.e., that

$$(x_1^s, \ldots, x_d^s) :_R (x_1^t, \ldots, x_d^t) = (x_1^s, \ldots, x_d^s) + y^{s-t}.$$

But

$$(x_1^s, \ldots, x_d^s) :_R (x_1^t, \ldots, x_d^t) = \bigcap_j (x_1, \ldots, x_s) :_R x_j^t = \bigcap_j \big((x_j^{s-t}) + I_s\big),$$

by part (b) of the first Proposition. The result we need now follows from part (f) of the first Proposition.

Thus, every endomorphism of E stabilizes the image of R_t for all t. Any endomorphism of R_t is evidently given by multiplication by a unique element of R_t, and so $\mathrm{Hom}_R(E, E)$ may be identified with $\varprojlim_t R_t \cong \widehat{R} \cong R$, as required. $\square$

Theorem. *Let (R, m, K) be a regular local ring and let $x_1, \ldots, x_d$ be a regular system of parameters in R. Let $I_t = (x_1^t, \ldots, x_d^t)R$. Let M be a finitely generated R-module, and $R \hookrightarrow M$ an injection such that $1 \mapsto u \in M$. Then $R \to M$ splits if and only if $I_tM \cap Ru = I_tu$ for every positive integer t.*

In particular if $R \subseteq S = M$ is an R-algebra, $R \to S$ splits if and only if $I_tS \cap R = I_t$ for every positive integer t. Hence, $R \to S$ splits if and only if $x_1^{t-1} \cdots x_d^{t-1} \notin I_tS$ for all t.

Proof. Let $\phi : M \to R$ be a splitting and let I be any ideal of R. Suppose that $ru \in IM$. Applying ϕ, we get that $r\phi(u) \in IR = I$, so that $IM \cap Ru = Iu$ for every ideal I of R. This shows that the stated condition is necessary for splitting.

$R \hookrightarrow M$ splits if and only if the induced map $\mathrm{Hom}_R(M, R) \to \mathrm{Hom}_R(R, R)$ is onto, and this is unaffected by completion, since Hom commutes with flat base change for Noetherian modules over a Noetherian ring and $\widehat{R}$ is faithfully flat over R. Thus, we may assume that R is complete without loss of generality. Note also that $R/I_t \cong \widehat{R}/I_t\widehat{R}$ for all t, and that $M/I_tM \cong \widehat{M}/I_t\widehat{M}$ for all t.

Since $R/I_t \to M/I_tM \cong (R/I_t) \otimes_R M$ is injective for all t, we may take a direct limit and obtain that $E \to E \otimes_R M \cong M \otimes_R E$ is injective, where E is constructed as in part (c) of the preceding Theorem and is injective over R. Applying $\mathrm{Hom}_R(\_, E)$ we find that the map $\mathrm{Hom}_R(M \otimes_R E,\, E) \to \mathrm{Hom}_R(E,\, E)$ is surjective, and by the adjointness of tensor and Hom, the left hand module may be identified with $\mathrm{Hom}_R\bigl(M,\, \mathrm{Hom}_R(E,\, E)\bigr)$. By part (d) of the preceding Theorem, $R \to \mathrm{Hom}_R(E,\, E)$ is an isomorphism, and so we have that the map $\mathrm{Hom}_R(M,\, R) \to \mathrm{Hom}_R(R,\, R)$ is surjective, as required.

The very last statement follows because $x_1^{t-1} \cdots x_d^{t-1}$ generates the socle in R/I_t, and so every ideal of R strictly larger than I_t contains $x_1^{t-1} \cdots x_d^{t-1}$. Thus, if $I_tS \cap R$ is strictly larger than I_t, it must contain $x_1^{t-1} \cdots x_d^{t-1}$: see part (a) of the preceding Theorem. $\square$

Conjecture (monomial conjecture). *Let $x_1, \, \ldots \, , x_d$ be be elements of a Noetherian ring R that generate an ideal of height d. Then for every positive integer t,*

$$x_1^{t-1} \, \cdots \, x_d^{t-1} \notin (x_1^t, \, \ldots \, , x_d^t)R.$$

If one has a counterexample, one can always localize at a minimal prime of $(x_1, \, \ldots \, , x_d)R$ of height d, and so obtain a counterexample in a local ring of dimension d in which $x_1, \, \ldots \, , x_d$ is a system of parameters. One can then complete, and so get a counter-example in a complete local ring. One can also kill a minimal prime so as to get a new counter-example in a complete local domain for which $x_1, \, \ldots \, , x_d$ is a system of parameters. We shall not completely prove the following result: for one of the implications in mixed characteristic we give a reference.

Theorem. *For local domains of a given characteristic (where this may refer to mixed characteristic p) and a given dimension d, the direct summand conjecture for regular local rings of that dimension and characteristic is equivalent to the monomial conjecture for systems of parameters of local rings of that dimension and characteristic.*

Proof. Assume the monomial conjecture in that dimension and characteristic. It suffices to prove the direct summand conjecture for complete regular local rings R of that dimension and characteristic and module-finite extension domains S of R. Let $x_1, \, \ldots \, , x_d$ be a regular system of parameters for R. Then it is also a system of parameters for S. The monomial conjecture applied to $x_1, \, \ldots \, , x_d$ and S shows that R is a direct summand of S, by the last statement of the preceding Theorem.

Now let S be a complete equicharacteristic domain and $x_1, \, \ldots \, , x_d$ a system of parameters for S. Let K be a coefficient field for S. Let $R = K[[x_1, \, \ldots \, , x_d]] \subseteq S$: R is regular and S is module-finite over R. The direct summand conjecture for R shows that R to S splits, and the fact that

$$x_1^{t-1} \, \cdots \, x_d^{t-1} \notin (x_1^t, \, \ldots \, , x_d^t)S$$

now follows from the final statement of the preceding Theorem.

For the mixed characteristic case of this result we refer the reader to [M. Hochster, *The direct summand conjecture and canonical elements in local cohomology modules*, J. of Algebra **84** (1983), 503–553]. $\square$

We have already established the existence of big Cohen-Macaulay modules in equal characteristic. This immediately yields the monomial conjecture and, hence, the direct summand conjecture.

Theorem. *The monomial conjecture holds for every local ring that has a big Cohen-Macaulay module. In fact, the monomial conjecture holds for every sequence of elements* $x_1, \ldots, x_d$ *that is a regular sequence on some module* M.

Hence the monomial conjecture holds for Noetherian rings that contain a field, and a regular ring that contains a field is a direct summand of every module-finite extension algebra.

Proof. First note that if $x_1, \ldots, x_d$ is a regular sequence on M, then

$$(x_1^t, \ldots, x_d^t)M :_M x_1^{t-1} \cdots x_d^{t-1} = (x_1, \ldots, x_d)M$$

(by part (b) of the first Proposition), and $(x_1, \ldots, x_d)M \neq M$ by the definition of a regular sequence. Thus,

$$(x_1^{t-1} \cdots x_d^{t-1})M \notin (x_1^t, \ldots, x_d^t)M,$$

and therefore

$$x_1^{t-1} \cdots x_d^{t-1} \notin (x_1^t, \ldots, x_d^t)R.$$

This proves the statement in the second sentence of the Theorem, and the statement in the first sentence is then immediate. The monomial conjecture for equicharacteristic rings follows because it reduces to the local case, and we have shown the existence of big Cohen-Macaulay modules in the local case for equicharacteristic rings. The final result is immediate from the preceding Theorem. $\square$

We conclude with some discussion of the notion of the *superheight* of an ideal I of a Noetherian ring R. If I is a proper ideal of R, we define the *superheight* of I as the supremum of heights of ideals IS, where $R \to S$ is a map of Noetherian rings such that IS is a proper ideal of S. (By convention, the height of the unit ideal is $+\infty$.) We want to make several observations about this notion.

First, the Krull height theorem is the same as the statement the the superheight of $\mathfrak{A} = (X_1, \ldots, X_d)$ in the polynomial ring $\mathbb{Z}[X_1, \ldots, X_d]$ is d, since the expansions $\mathfrak{A}$ to various choices of S are the same as the ideals with at most d generators in the various Noetherian rings S. It follows that the superheight of any ideal is at most the number of generators of that ideal. Since two ideals with the same radical have the same height, they also have the same superheight, and the superheight of I is bounded by the least number of generators of an ideal J such that J and I have the same radical.

If $R \to S$ is such that the height of IS is the superheight of I, this remains true when we local IS at a suitable minimal prime of IS, complete, and kill a suitable minimal prime. Thus, in the definition of superheight, it suffices to allow S to run through complete local domains to which R maps such that IS is primary to the maximal ideal of S.

Consider the following example: let K be a field, let $X_1, \ldots, X_n$ and $Y_1, \ldots, Y_n$ be indeterminates over K, let $T = K[X_1, \ldots, X_n, Y_1, \ldots, Y_n]$, and let P be the ideal generated by the size 2 minors of the matrix

$$\begin{pmatrix} X_1 & \cdots & X_n \\ Y_1 & \cdots & Y_n \end{pmatrix}.$$

Let $R = T/P$. It is known that R is a domain of dimension $n+1$. (See **3.** in Problem Set #4, Math 614, Fall 2003, and its solution, where it is shown that the corresponding algebraic set is irreducible: this at least shows that the radical of the ideal is prime. The open set where the first row is not 0 is dense. One sees that the dimension is $n+1$ because the matrix determines and is determined by the nonzero first row and the scalar whose product with the first row gives the second row. For a complete treatment of varieties of this type, see [M. Hochster and J. A. Eagon, *Cohen-Macaulay rings, invariant theory, and the generic perfection of determinantal loci*, Amer. J. Math. **93** (1971), 1020–1058].) Let $I = (X_1, \ldots, X_n)R$. Then $R/I = K[Y_1, \ldots, Y_n]$ has dimension n, and so I is a height one prime ideal of R. Let $J = (Y_1, \ldots, Y_n)R$. Then $S = R/J \cong K[X_1, \ldots, X_n]$, a polynomial ring in n variables over K. The expansion of I to S has height n. Thus, even though I has height one, its superheight is n.

The problem of determining the superheight of an ideal is extremely difficult. Let Λ denote either a field or the integers $\mathbb{Z}$. Let

$$R_{d,t} = \Lambda[X_1, \ldots, X_d, Y_1, \ldots, Y_d]/F_{t,d},$$

where

$$F_{t,d} = X_1^{t-1} \cdots X_d^{t-1} - \sum_{j=1}^{d} Y_j X_j^t.$$

Let $I = (X_1, \ldots, X_d)R$. Then the monomial conjecture is precisely equivalent to the statement that, when $\Lambda = \mathbb{Z}$, the superheight of I is $d-1$. It is easy to see that it must be $d-1$ or d. But the question of which it is remains open if $\Lambda = \mathbb{Z}$ and $d \geq 4$. The direct summand conjecture is known to hold for rings containing a field, and so the superheight is $d-1$ if Λ is a field, but this is not obvious. The direct summand conjecture and even the existence of big Cohen-Macaulay algebras are known in mixed characteristic in dimension at most three (see [R. Heitmann, *The direct summand conjecture in dimension three*, Annals of Math. (2) **156** (2002) 695–712] and [M. Hochster, *Big Cohen-Macaulay algebras in dimension three via Heitmann's theorem*, J. Algebra **254** (2002) 395–408]. Thus the superheight is known to be $d-1$ when $\Lambda = \mathbb{Z}$ if $d \leq 3$.

The following result, which we will prove in seminar next semester, is essentially due to Serre:

Theorem. *If P is a prime ideal of a regular ring, the superheight of P is equal to the height of P.*

One can reduce to the case where R is regular local. Then, when S has the form R/Q, one needs to see that the height of $P(R/Q)$ is at most the height of P. This is Serre's result, in [J.-P. Serre, *Algèbre Locale • Multiplicités*, Lecture Notes in Math. **11**, Springer-Verlag, Berlin • Heidelberg • New York, 1965]. The general case can be reduced to this one.

Note that this Theorem is a vast generalization of the Krull height theorem, which is the case where $R = \mathbb{Z}[X_1, \ldots, X_d]$ and $P = (X_1, \ldots, X_d)$.