Math 711, Fall 2004 **Problem Set #1**

Due: Wednesday, October 13

**1.** Let $(R, m, K)$ be a local ring in which the integer $n$ is invertible and let $u \in m$ be any element.

(a) Let $S = R[z]/(z^n - 1 - u)$. Show that $R \to S$ is étale.

(b) Is $R[z]/(z^2 - u)$ ever étale over $R$? Prove your answer.

**2.** Let $R \to S \to T$ be ring homomorphisms. Prove carefully, preferably using appropriate universal properties, that there is an exact sequence $T \otimes_S \Omega_{S/R} \to \Omega_{T/R} \to \Omega_{T/S} \to 0$. Note that there is no claim of injectivity for the first map.

**3.** Let $F : K[x_1, \dots, x_n] \to K[x_1, \dots, x_n]$ be a $K$-endomorphism that maps $x_j$ to $f_j$, $1 \le j \le n$. Prove that $F$ is étale if and only if $\det\big(\frac{\partial f_j}{\partial x_i}\big)$ is an element of $K - \{0\}$.

**4.** Let $h : R \to S$ be étale, where both are $A$-algebras. Prove that for every $A$-derivation $D : R \to M$, where $M$ is an $R$-module, there is a unique derivation $\widetilde{D} : S \to S \otimes_R M$ such that for all $f \in R$, $\widetilde{D}\big(h(f)\big) = 1 \otimes D(f)$. Briefly put, $D$ extends uniquely to a derivation $S \to S \otimes_R M$.

**5.** Let $R$ be a Noetherian algebra over $\mathbb{Z}/p\mathbb{Z}$. Prove or disprove the following statement: $R$ is formally unramified over $\mathbb{Z}/p\mathbb{Z}$ if and only if it is a finite product of perfect fields.