Math 711, Fall 2004 **Problem Set #2**
Due: Wednesday, October 27

1. Let R be a ring and let S and T be R algebras both of which are (a) finitely presented, (b) formally unramified, (c) formally smooth, (d) formally étale, (e) smooth, (f) unramified, or (g) étale. Prove that $S \otimes_R T$ has that property as well.

2. Let S be an R-algebra. Prove or disprove the following statement: S is formally smooth (respectively, formally unramified) over R if and only if for every integer $n \geq 2$, every R-algebra T, and every ideal J of T such that $J^n = 0$, every R-algebra homomorphism $S \to T/J$ has at least one (respectively, at most one) lifting to an R-algebra homomorphism $S \to T$.

3. Let $1 < r \leq s$ be integers and let $X = \bigl(x_{ij}\bigr)$ be an $r \times s$ matrix of indeterminates over K. Let $T = K[X]$ be the polynomial ring generated by the x_{ij}, let t be a integer, $0 < t < r$, and let $P = I_{t+1}(X)$, the ideal generated by the size $t+1$ minors of X. You may assume that P is a prime ideal such that $S = T/P$ has dimension $st + rt - t^2$ (this is true). Let Q be a prime ideal of S. Prove that S_Q fails to be formally smooth over K if and only if Q contains the image of $I_t(X)$.

4. Let R be a ring.

(a) Let J be a finitely generated ideal of R. Prove that $J = J^2$ if and only if J is generated by an idempotent.

(b) Prove that a finitely generated algebra S over R is formally unramified if and only if $S \otimes_R S$ is R-isomorphic to with a product ring $T \times S$ in such a way that when one composes the isomorphism with the product projection $\pi_2 : T \times S \twoheadrightarrow S$ one obtains the usual R-algebra homomorphism $S \otimes_R S \twoheadrightarrow S$ (sending $s \otimes s' \mapsto ss'$). In other words, $S \otimes_R S \twoheadrightarrow S$ is, up to isomorphism, a product projection.

5. Let K be an algebraically closed field. Let S be a given finite subset of K^2. Show that there exist polynomials $f(x,\, y),\, g(x,\, y) \in K[x,\, y]$ such that for $c,\, d \in K$ the simultaneous equations $f(x,\, y) = c$, $g(x,\, y) = d$ have at least one solution if and only if $(c,\, d) \notin S$. (If you cannot solve the general problem begin by solving the problem when $S = \{(0,\, 0)\}$.)