Math 711, Fall 2004 **Problem Set #4**
Due: Wednesday, November 24

1. Let Λ be a directed partially ordered set and let $\{S_\lambda\}_{\lambda \in \Lambda}$ be a direct limit system of algebras each of which is formally étale over the ring R. Prove that $S = \varinjlim_\lambda S_\lambda$ is formally étale over R.

2. Show that if L is a separable algebraic extension of K then L is formally smooth over K. Also show that if L has a separating transcendence basis over K then L is formally smooth over K.

3. Let K be a field of positive characteristic p, and let x be an indeterminate over K. Let L the field obtained by adjoining x^{1/p^n} to $K(x)$ for all positive integers n.

(a) Show that if K is perfect, then L is formally smooth over K.

(b) If K is not perfect, must L be formally smooth over K?

4. Let (R, P, K) be a Noetherian local ring that is an approximation ring. Show that if R is reduced then $\widehat{R}$ is reduced. Show that if R is a domain, then $\widehat{R}$ is a domain.

EXTRA CREDIT If R is an approximation ring and R is normal, is $\widehat{R}$ necessarily normal?

5. Let X and Y be varieties in $\mathbb{A}^n_{\mathbb{C}}$ that have the origin as an isolated point of intersection.

(a) Suppose that $n = 2$, that $X = V(f)$, and that $Y = V(g)$, where f and g are irreducible. Show that the intersection multiplicity of X and Y at the origin is $\ell\big(\mathbb{C}[[x, y]]/(f, g)\big)$ using the characterization in terms of Tor.

(b) Let $n = 4$, and let X be $V(P)$ where P is the kernel of the $\mathbb{C}$-algebra surjection $K[x_1, \ldots, x_4] \twoheadrightarrow \mathbb{C}[x^2, x^3, xy, y] \subseteq \mathbb{C}[x, y]$ such that $x_1 \mapsto x^2$, $x_2 \mapsto x^3$, $x_3 \mapsto xy$, and $x_4 \mapsto y$. You may assume that P is $I_2\begin{pmatrix} x_1 & x_2 & x_3 & x_4 \\ x_2 & x_1^2 & x_1x_4 & x_3 \end{pmatrix}$. Let $Y = V(Q)$ where $Q = (x_2, x_4)$. Determine the intersection multiplicity of X and Y at the origin, and show that it is not $\ell\big(\mathbb{C}[[x_1, x_2, x_3, x_4]]/(P + Q)\big)$.

EXTRA CREDIT Let R be a Noetherian ring that is formally smooth over a perfect field K. Prove that R is regular. (Suggestion: reduce to the local case, (R, P). One has a coefficient field $L \subseteq R/P^2$ (which is complete) with $K \subseteq L$. Let $x_1, \ldots, x_d$ be a minimal set of generators of P. Then we have a surjection $R \twoheadrightarrow R/P^2 \cong L[X_1, \ldots, X_d]/m^2$, where $X_1, \ldots, X_d$ are indeterminates and $m = (X_1, \ldots, X_d)$. Use that R is formally smooth over K to show that this lifts to a map $R \twoheadrightarrow L[X_1, \ldots, X_d]/m^{n+1}$ for all n. Use this to show that $\ell(R/P^{n+1}) \geq \ell(L[X_1, \ldots, X_d]/m^{n+1})$. Conclude that $\dim(R) \geq d$.)

[The condition that K be perfect can removed: in the general case one needs to prove that every field K is formally smooth over the prime field it contains, which is perfect.]