Math 711, Fall 2004 **Problem Set #5**
Due: Monday, December 13

1. Let S be a finitely generated algebra over a field K. Prove that S is separable over K if and only if S is reduced and for every mininal prime P of S, S/P is separable over K.

2. (a) Let S be a Henselian local domain and let T be a domain that is a module-finite extension of S. Prove that T is local.

(b) Let (R, m, K) be a local domain and suppose that R^{h} is a domain. Prove that every ring obtained by adjoining finitely many integral elements of the fraction field of R to R is local.

3. Let (R, m, K) be a local ring that is an approximation ring. Suppose that R is a UFD. Prove that $\widehat{R}$ is a UFD.

4. Prove or disprove the following assertion: if L is a finitely generated field extension of the field K of transcendence degree n over K, then $u_1, \ldots, u_n$ is a separating transcendence basis for L over K if and only if $du_1, \ldots, du_n$ is an L-vector space basis for $\Omega_{L/K}$.

5. Prove that every field extension L of a perfect field K is formally smooth. (Suggestions: in characteristic zero, any transcendence basis is separating. In characteristic p, choose a p-base Λ for L over L^p and suppose one has a K-map $f : L \to T/J$ where $J^2 = 0$. Show that one gets a K-homomorphism $L^p \to T$ by letting its value on c^p be t^p, where $t \in T$ is any lifting of of $f(c) \in T/J$. This is independent of the choice of t, and lifts the restriction of f to L^p. Now make use as well of a choice of liftings for the elements $f(\lambda)$, $\lambda \in \Lambda$.) [This completes the proof that a Noetherian ring formally smooth over a field is regular.]

EXTRA CREDIT Let S be a local ring of a finitely generated R-algebra, where (R, m, K) is local. We know that if S is a pointed étale extension of R then the induced map $\widehat{R} \to \widehat{S}$ is an isomorphism.

If $R \to S$ induces an isomorphism $\widehat{R} \to \widehat{S}$ then:

(a) Must S be étale over R?
(b) Must S be flat over R?
(c) If R is regular must S be étale over R?
(d) If If R is a local ring at a maximal ideal of an algebra finitely generated over the complex numbers $\mathbb{C}$, must S be étale over R?