Commutative Algebra Seminar: Lecture of January 12, 2006

It was shown in Math 711 last semester that the following are equivalent conjectures (these statements are known in the equal characteristic case and in dimension at most 3):

(1) Every regular Noetherian ring is a direct summand of every module-finite extension ring. (The direct summand conjecture.)

(2) If $x_1, \ldots, x_d$ are elements generating an ideal of height d in a Noetherian ring R, then for every $t \in \mathbb{N}$, $(x_1 \cdots x_d)^t \notin (x_1^{t+1}, \ldots, x_d^{t+1})R$. (The monomial conjecture.)

(3) If $x_1, \ldots, x_d$ is a system of parameters for a local ring (R, m, K), and one has a lifting of the obvious quotient map $R/(x_1, \ldots, x_d)R \twoheadrightarrow K$ to a map from the Koszul complex $\mathcal{K}_\bullet(x_1, \ldots, x_d;\, R)$ to a truncated free resolution of K, say $G_\bullet$, where $G_\bullet$ is

$$0 \to \mathrm{syz}^d(K) \to G_{d-1} \to \cdots \to G_1 \to G_0 \to 0$$

(here, the complex is acyclic with $H_0(G_\bullet) = K$ and the G_j are free), then the induced map $R = \mathcal{K}_d(x_1, \ldots, x_d;\, R) \to \mathrm{syz}^d(K)$ is not zero. (The canonical element conjecture.)

All three statements reduce to the case where R is local and complete, or even a complete local domain. E.g., in the case of (2) we may localize at a minimal prime of $(x_1, \ldots, x_d)R$ of height d and so reduce to the case where $x_1, \ldots, x_d$ is a system of parameters of a local ring (R, m, K), which we may then complete.

In (3), the statement can be strengthened. Not only is the map nonzero, but it has nonzero image in $M/(x_1, \ldots, x_d)M$, where $M = \mathrm{syz}^d(K)$ is the dth module of syzygies being used. Moreover, by varying the system of parameters, one sees that the image η of 1 in $H^d_m(M)$, which may be thought of as the direct limit of all the modules $M/(x_1, \ldots, x_d)M$ as the system of parameters $x_1, \ldots, x_d$ varies, is nonzero. This image is called the *canonical element* in $H^d_m(M)$. The canonical element conjecture may be viewed as stating that the canonical element in $H^d_m\bigl(\mathrm{syz}^d(K)\bigr)$ is not zero.

It was also shown that the direct summand conjecture reduces to the case where the regular ring is a formal power series ring over a mixed characteristic complete Noetherian discrete valuation ring V with maximal ideal pV, where p is the characteristic of the residue class field: in particular, one need not consider ramified regular local rings.

In addition to the above conjectures, the following was considered:

(4) Let (R, m, K) be a local ring and let $G_\bullet$ be a finite complex of finitely generated free modules, say

$$0 \to G_n \to \cdots \to G_1 \to G_0 \to 0$$

such that $H_0(G_\bullet) \neq 0$ has a minimal generator that is killed by a power of m and $H_i(G_\bullet)$ has finite length for $i > 0$. Then $\dim(R) \le n$. (The improved new intersection theorem.)

This statement was used by Evans and Griffith in the proof of their syzygy theorem.

We showed that (3) $\Rightarrow$ (4). The converse, (4) $\Rightarrow$ (3), is also true. In fact, the four statements are equivalent for local rings of given dimension and characteristic (where the characteristic might be specified as "mixed characteristic p" for a specific prime number p). It was shown that all four statements follow from the existence of big Cohen-Macaulay modules.

We may weaken (4) by strengthening the hypothesis and assuming that $H_0(G_\bullet) \neq 0$ has finite length. For convenience of reference, we make this explicit.

(a) Let (R, m, K) be a local ring and let $G_\bullet$ be a finite complex of finitely generated free modules, say

$$0 \to G_n \to \cdots G_1 \to G_0 \to 0$$

such that $H_0(G_\bullet) \neq 0$ and $H_i(G_\bullet)$ has finite length for $i \geq 0$. Then $\dim(R) \leq n$. (New intersection theorem.)

This statement, the *new intersection theorem*, does seem to be weaker than (1) – (4). It was formulated independently by Peskine-Szipro and Paul Roberts.

An even weaker version is:

(b) If (R, m, K) is local, $M, N \neq 0$ are finitely generated R-modules, and $M \otimes_R N$ has finite length, then $\dim(N) \leq \mathrm{pd}_R M$. (The Peskine-Szpiro intersection theorem.)

This was formulated, and proved in characteristic $p > 0$ as well as in many cases in equal characteristic 0, by Peskine and Szpiro. We proved, following Peskine-Szpiro, that (b) implies

(c) If R is local, $M \neq 0$ is finitely generated and $\mathrm{pd}_R M < \infty$, every zerodivisor in R is a zerodivisor on M. (M. Auslander's zerodivisor conjecture.)

We also proved, again following Peskine-Szpiro, that (b) implies

(d) If R is local and $N \neq 0$ is a finitely generated module of finite injective dimension, then R is Cohen-Macaulay. (Bass's question.)

Paul Roberts has proved that the new intersection theorem holds in all dimensions, even in mixed characteristic, and, hence, so do (b), (c), and (d). In these lectures we shall give his proof and discuss further the relationships among (1), (2), (3), and (4), including other statements that are equivalent to these.

In order to give his proof, we need results from the theory of local Chern characters. This theory was developed by Baum, Fulton and MacPherson and is treated in [W. Fulton, *Intersection Theory*, Springer-Verlag, Berlin, 1984]. See especially Chapter 18, with the notion of dimension modified as in Chapter 20. Other references for this material include the expository paper [P. Roberts, *Intersection Theorems*, in Commutative Algebra (M. Hochster, C. Huneke, J.D. Sally, editors) MSRI Publ. **15**, Springer-Verlag, New York, 1989, pp. 417–436], [P. Roberts, *Multiplicities and Chern classes in local algebra*, Cambridge Tracts in Mathematics **133**, Cambridge University Press, Cambridge, England, 1998], and my review of this book for the Bulletin of the A.M.S., which is available from the expository manuscripts section of my web page.

Let A be a local ring, which we shall assume for simplicity is a homomorphic image of regular local ring. Since all of the conjectures that we are interested in here reduce to the complete local case, there is no essential loss of generality from this restriction. Let $X = \mathrm{Spec}\,(A)$. We denote by $Z_i(X)$ the free abelian group whose free generators are the reduced and irreducible closed subschemes of X of dimension i. These are in bijective correspondence with the prime ideals of A such that $\dim\,(A/P) = i$, with $\mathrm{Spec}\,(A/P) \hookrightarrow X$ thought of as a closed subscheme, and we write $[A/P]$ for the element of $Z_i(X)$ corresponding to P.

If W is a local domain of dimension 1, and x is an element of $W - \{0\}$, we define the *order* of x in W, $\mathrm{ord}_W(x)$, to be $\ell_W(W/xW)$, where $\ell_W(\_)$ denotes length as as W-module. Note that if $x, y \in W - \{0\}$, we have a short exact sequence

$$0 \to W/yW \to W/xyW \to W/xW \to 0,$$

since $xW/xyW \cong W/yW$, and it follows that

$$\mathrm{ord}_W(xy) = \mathrm{ord}_W(x) + \mathrm{ord}_W(y)$$

for all $x, y \in W - \{0\}$. It follows that $\mathrm{ord}_W(\_)$ extends uniquely to a group homomorphism $\mathrm{frac}\,(W) - \{0\} \to \mathbb{Z}$.

When W happens to be a discrete valuation ring, this agrees with the usual notion of order of an element in a DVR.

If Q is a prime ideal with $\dim\,(A/Q) = i + 1$ and $x \in A/Q$, we define $\mathrm{div}\,(x) \in Z_i(X)$ to be $\sum_P \mathrm{ord}_{(A/Q)_P}(x)[A/P]$, where P runs through primes such that $\dim\,(A/P) = i$ and $P \supseteq Q + xA$. These are in bijective correspondence with the minimal primes of x in A/Q, and so there are only finitely many.

We shall write $A_i(X)$ for the abelian group that is the quotient of $Z_i(X)$ by the subgroup spanned by all elements of the form $\mathrm{div}\,(x)$, and we shall write $\mathcal{A}_i(X)$ for $\mathbb{Q} \otimes_{\mathbb{Z}} A_i(X)$. Some of the constructions we need require denominators, and so we shall usually work with the groups $\mathcal{A}_i(X)$. If $\dim\,(A) = d$, we write $\mathcal{A}_*(X)$ for

$$\bigoplus_{i=0}^{d} \mathcal{A}_i(X).$$

Let $F_\bullet$ be a bounded complex of finitely generated free A-modules. Let Z denote the *support* of $F_\bullet$, the reduced closed subscheme of X whose points correspond to all primes P such that the localization of $F_\bullet$ at P is not exact. That is, Z is the support of the total homology module $\bigoplus_i H_i(F_\bullet)$. The *local Chern character* $\mathrm{ch}(F_\bullet)$ of $F_\bullet$ is a sum of operators

$$\mathrm{ch}(F_\bullet) = \bigoplus_{i=0}^{d} \mathrm{ch}_i(F_\bullet),$$

where $\mathrm{ch}_i(F_\bullet)$ gives a map of graded abelian groups of degree $-i$ from $\mathcal{A}_*(X) \to \mathcal{A}_*(Z)$. That is, for all n, $\mathrm{ch}_i(F_\bullet) : \mathcal{A}_n(X) \to \mathcal{A}_{n-i}(Z)$.

If Z' is a closed set containing Z we have a map $\mathcal{A}_*(Z) \to \mathcal{A}_*(Z')$, and so it is also possible to consider values for these operators in the Chow group of any closed subscheme containing the support of the homology. Moreover, if $F_\bullet$ is a complex of free modules on X, $\mathrm{ch}(F_\bullet)$ acts on $\mathcal{A}_*(Y)$ for every closed subscheme Y of X, for we may restrict the complex $F_\bullet$ to get a bounded free complex on Y.

The operators $\mathrm{ch}_i(F_\bullet)$ are extraordinarily well-behaved. We shall not go into great detail here, but we mention that they are compatible with proper push-forward and also with intersection with Cartier divisors: the corresponds to killing a nonzerodivisor in the ring.

Moreover, for bounded free complexes $F_\bullet$, $G_\bullet$, $H_\bullet$ with supports V, W, and Z, respectively:

(i) One has additivity: if $0 \to F_\bullet \to G_\bullet \to H_\bullet \to 0$ is exact then

$$\mathrm{ch}(G_\bullet) = \mathrm{ch}(F_\bullet) + \mathrm{ch}(H_\bullet)$$

in $\mathcal{A}_*(V \cup Z)$.

(ii) One has a multiplicative property:

$$\mathrm{ch}(F_\bullet \otimes_A G_\bullet) = \mathrm{ch}(F_\bullet)\mathrm{ch}(G_\bullet)$$

in $\mathcal{A}_*(V \cap W)$.

(iii) One has commutativity: for all i and j,

$$\mathrm{ch}_i(F_\bullet)\mathrm{ch}_j(G_\bullet) = \mathrm{ch}_j(G_\bullet)\mathrm{ch}_i(F_\bullet)$$

in $\mathcal{A}_*(V \cap W)$.

Note that the tensor product of two complexes is exact whenever one of them is, by the spectral sequence for a double complex. It follows that the support of $F_\bullet \otimes_A G_\bullet$ is contained in the intersection of the supports of $F_\bullet$ and $G_\bullet$, so that the left hand side in (ii) can be taken as an element of $\mathcal{A}_*(V \cap W)$. In both (ii) and (iii), observe that the value of $\mathrm{ch}_j(G_\bullet)$ can be taken in $\mathcal{A}_*(W)$, and that the restriction of $F_\bullet$ to W has support contained in $V \cap W$, so that the value of the right hand side can be also be taken in $\mathcal{A}_*(V \cap W)$. A similar remark applies to $\mathrm{ch}_j(G_\bullet)\mathrm{ch}_i(F_\bullet)$.