Commutative Algebra Seminar: Lecture of January 19, 2006

We want to mention two more properties of local Chern classes. Recall first that A is a local ring that is a homomorphic image of a regular local ring and that $X = \mathrm{Spec}\,(A)$. We write $\mathcal{A}_*(A)$ for the Chow group of A with rational coefficients. Recall also that $F_\bullet$ denotes a bounded complex of finitely generated free modules.

(iv) If $F_\bullet^\vee = \mathrm{Hom}_A(F_\bullet,\, A)$ is the dual of $F_\bullet$, then $\mathrm{ch}_i(F_\bullet^\vee) = (-1)^i\mathrm{ch}(F_\bullet)$.

A finitely generated module M over A has a Todd class $\tau(M)$ in $\mathcal{A}_*\bigl(\mathrm{Supp}\,(M)\bigr)$. Let R be a regular local ring that maps onto A and let $F_\bullet$ be a finite free resolution of M over R. Then $\tau(M)$ is the result of applying $\mathrm{ch}(F_\bullet)$ to $[R]$ in $\mathcal{A}_*\bigl(\mathrm{Spec}\,(R)\bigr)$. We shall discuss the notion of Todd classes in more detail and greater generality later.

(v) One has a local Riemann-Roch formula. Let z be the closed point of X. Then $\mathcal{A}_*(z) = \mathcal{A}_0(z) = \mathbb{Q}z$, which we identify with $\mathbb{Q}$, mapping $z \mapsto 1$. If $F_\bullet$ has homology of finite length, then for every finitely generated A-module M,

$$\chi(F_\bullet \otimes_A M) = \mathrm{ch}(F_*)\bigl(\tau(M)\bigr).$$

Before proceeding further with applications of this theory, we want to explain a bit more how one gets these operators, although we shall still be omitting a great deal.

First, we want to generalize the construction of the Chow group to schemes of finite type over A. As mentioned last time, we want to do this so that localization of a domain at one element does not effect the dimension.

The schemes that we shall be considering are finite unions of open affines, each of which has the form $Y = \mathrm{Spec}\,(B)$ for some finitely generated A-algebra B. When B is a domain, the map $A \to B$ factors uniquely as

$$A \twoheadrightarrow A/P \hookrightarrow B,$$

where A/P is a domain. In this case, we define the dimension of B, and of $Y = \mathrm{Spec}\,(B)$, to be the sum of $\dim\,(A)$ and the transcendence degree of the fraction field of B over the fraction field of A/P. Clearly, this does not change when we localize at one element of B. We shall denote this dimension as $\mathtt{dim}(B)$ or $\mathtt{dim}(Y)$. Note that $\dim\,(A/I) = \mathtt{dim}(A/I)$ for every ideal I. When B is not necessarily a domain, we let $\mathtt{dim}(B) = \mathtt{dim}(Y)$ be the largest value of $\mathtt{dim}(B/P)$ as P runs through the minimal primes of B. Finally, when Y is a scheme of finite type over A, it is covered by finitely many open affines, and we let $\mathtt{dim}(Y)$ be the largest dimension, in our new sense, of any of these. This is independent of the choice of cover. When Y is irreducible, $\mathtt{dim}(Y)$ is the same as $\mathtt{dim}(U)$ for every non-empty open affine U. (This notion differs by a translation from the one that is used in Chapter 20 Fulton's book: there, one considers schemes of finite type over a regular local ring, and the notion of dimension is set up so that the regular local ring has dimension 0.

It is more convenient here to have the new notion agree with Krull dimension on closed subschemes of Spec (A).)

If Y is a reduced and irreducible scheme, then the ring of sections on any non-empty open affine U has the same field of fractions as does any other, U'. This field is called the *function field* of Y, and we denote it $\mathcal{R}(Y)$.

The Chow group $A_*(Y)$ of a scheme Y of finite type over A is $\bigoplus_i A_i(Y)$, where $A_i(Y)$ is obtained as a quotient of the free abelian group $Z_i(Y)$ whose free basis is the set of reduced and irreducible subschemes of dimension i. The subgroup one kills is spanned by certain divisors, but the description is slightly more complicated than it was earlier. Given a reduced and irreducible subscheme Z of dimension $i+1$ and a nonzero element $f \in \mathcal{R}(Z)$, we can define $\mathrm{div}\,(f)$ as follows: for each closed irreducible V contained in Z, we shall define $\mathrm{ord}_V(f)$, and then

$$\mathrm{div}\,(f) = \sum_{V \subseteq Z} \mathrm{ord}_V(f)[V],$$

where V runs through all closed reduced and irreducible subschemes of Z of dimension i, and we are using $[V]$ for the element represented by V in $Z_i(Y)$. (Later, we shall also use $[V]$ for the image of $[V]$ as above in $A_i(Y)$ or in $\mathcal{A}_i(Y)$.) Choose an open affine of Z that meets V, and replace V by its intersection V' with this open affine. Localize the ring of sections of this affine at the height one prime corresponding to V'. This gives a one-dimensional local domain W and f is a nonzero element of the fraction field of W. Thus, we have a value for $\mathrm{ord}_W(f)$ as before, and we define this to be the value of $\mathrm{ord}_V(f)$. We next show that given f, there are only finitely many V such that $\mathrm{ord}_V(f)$ is not 0.

To see this, take a cover of W by finitely many non-empty open affines. Each V that occurs with nonzero coefficient must meet one of these, and so we reduce to the case where Z is replaced by an open affine Spec (B), where B is a domain, and $f = b/c$ is a nonzero element of the fraction field of B, where $b, c \in B - \{0\}$. For every V, $\mathrm{ord}_V(f) = \mathrm{ord}_V(b) - \mathrm{ord}_V(c)$, and so we need only see that $\mathrm{ord}_V(b)$ has only finitely many nonzero coefficients when $b \in B - \{0\}$. But now the choices of V for which one has a nonzero coefficient correspond to the height one primes of B that contain b, and so there are only finitely many.

Consider the subgroup of $Z_i(Y)$ spanned by the divisors of all such f for all choices of Z as above. $A_i(Y)$ is the quotient of $Z_i(Y)$ by this subgroup, and

$$\mathcal{A}_i(Y) = \mathbb{Q} \otimes_{\mathbb{Z}} A_i(Y).$$

We thus have a graded abelian group

$$A_*(Y) = \bigoplus_i A_i(Y)$$

and graded $\mathbb{Q}$-vector space

$$\mathcal{A}_*(Y) = \bigoplus_i \mathcal{A}_i(Y)$$

as before.

Now suppose that $f : Y \to Z$ is a proper morphism. Thus, images of closed sets are closed, and this remains true after base change. (Recall also that projective morphisms are proper, and that blowing up gives a projective morphism.) We want to describe a degree preserving map, *proper push-forward*, $f_* : \mathcal{A}_i(Y) \to \mathcal{A}_i(Z)$ for all i. (This map is also define for integer coefficients instead of rational coefficients.) Let V be closed, reduced and irreducible of dimension i in Y. Its image V' in Z is then closed of dimension i or smaller. If its dimension is smaller than i, we define $f_*([V]) = 0$. If its dimension is i, then the restriction of f gives a surjection $V \twoheadrightarrow V'$, and there is an induced extension of function fields $\mathcal{R}(V') \hookrightarrow \mathcal{R}(V)$ that is finite algebraic. We define

$$f_*([V]) = [\mathcal{R}(V) : \mathcal{R}(V')][V']$$

in this case. Here $[V]$ and $[V']$ represent images in the respective Chow groups. Of course, one must check that this map is well-defined: we are not giving details here.

We next want to define an induced pull-back map for flat morphisms that have a well-defined relative dimension (this map is defined for integer as well as for rational coefficients). We say that a flat morphism $f : Y \to Z$ has *relative dimension* n if for every closed reduced and irreducible subscheme V of Z, $f^{-1}(V)$ has pure dimension equal to $\dim(V) + n$. Note that open immersions are flat of relative dimension 0. When f is flat of relative dimension n, we want to define $f^* : \mathcal{A}_i(Z) \to \mathcal{A}_{i+n}(Z)$ such that

$$f^*([V]) = [f^{-1}(V)] :$$

we need to define the right hand side when $f^{-1}(V)$ is a closed subscheme of pure dimension.