Commutative Algebra Seminar: Lecture of January 26, 2006

If Z is a closed subscheme of Y of pure dimension k, we can associate an element of $Z_k(Y)$, and, hence, of $A_k(Y)$ and $\mathcal{A}_k(Y)$, with Z: in all three cases, we denote this element $[Z]$. The element $[Z] \in Z_k(Y)$ will be an integer linear combination of the irreducible components V_i of Z. Let $V = V_i$ be one of these. The question that remains to be answered is what coefficient $\mu = \mu_i$ do we use for V. We simply use the length of the Artin local ring of the scheme Z at the point corresponding to V. We can describe this ring alternatively as follows. Choose an open affine U in Y that meets Z. Let B be the ring of sections on U, so that $U = \mathrm{Spec}\,(B)$. Let I be the ideal of B that defines $Z \cap U$ (I need not be radical), so that $Z \cap U = \mathrm{Spec}\,(B/I)$. Then $V \cap U$ corresponds to a minimal prime P of I, and the Artin local ring we want is B_P/IB_P. Then $[Z] = \sum_i \mu_i [V_i]$ in $Z_k(Y)$, in $A_k(Y)$, and in $\mathcal{A}_k(Y)$.

We can now make our definition of flat pull-back for a flat morphism $f : Y' \to Y$ of relative dimension n precise: if V is closed, reduced and irreducible of dimension i in Y, then $f^*([V]) = [f^{-1}(V)]$, where $f^{-1}(V)$ is the scheme-theoretic inverse image of V (obtained locally by expanding the defining ideal). Note that, by the definition of relative dimension, $f^{-1}(V)$ has pure dimension $\mathtt{dim}(V) + n$.

Proper push-forward and flat pull-back give functorial operations both for $A_*(\_)$ and $\mathcal{A}_*(\_)$. Proper push-forward preserves degree. Flat pull-back shifts degrees upwards by n, where n is the relative dimension. In the special case of an open immersion, degree is preserved.

We next want to discuss two notions of vector bundle. One of them is that a vector bundle on a scheme X is simply a locally free coherent sheaf of constant rank r on X. If X is connected, the rank is automatically constant. In the case of an affine scheme, this simply means that a vector bundle is given by a module locally free of rank r over the corresponding ring.

The second notion is that a vector bundle of rank r on X is a scheme E together with a projection map $\pi : X \to E$ satisfying the following condition: there is an open cover $\mathcal{U}$ of X such that on each $U \in \mathcal{U}$, $\pi^{-1}(U)$ is isomorphic with $U \otimes_{\mathbb{Z}} \mathrm{Spec}\,(\mathbb{Z}[x_1, \ldots, x_r]) = \mathbb{A}^r_U$ in such a way that the transition maps on overlaps are linear. This means that if we use the identifications $\mathbb{A}^r_U \cong \pi^{-1}(U)$ and $\pi^{-1}(V) \cong \mathbb{A}^r_V$, each restricted to $U \cap V$, to get an automorphism $\mathbb{A}^r_{U\cap V} \cong \mathbb{A}^r_{U\cap V}$, the automorphism is linear (on each open affine $W \subseteq U \cap V$, the corresponding automorphism of $B[x_1, \ldots, x_r]$, where B is the ring of sections of X on W, is a B-algebra map induced by a B-linear automorphism of the one-forms $B_1 = Bx_1 + \cdots + Bx_r$). The two notions correspond, up to isomorphism, by virtue of the following two constructions, which are mutually inverse up to isomorphism.

First, given a locally free sheaf $\mathcal{E}$ of rank r on X, we can form $\pi : E \to X$ so that over an open affine $U = \mathrm{Spec}\,(B)$ for which the $\mathcal{E}(U)$ is B-free, $\pi^{-1}(U) \cong \mathrm{Sym}_B\big(\mathcal{E}^\vee(U)\big)$. These schemes glue together correctly as U varies.

Second, given E, define a *section* of E on U to be a map $\sigma : U \to E|_U$ whose compostion with the restriction of π is the identity on U. When E is a vector bundle of rank r on X, the sections of E form a locally free coherent sheaf of rank r.

For example, when X is simply $\mathrm{Spec}\,(B)$ and the vector bundle corresponds to the module $G = B^r$, the scheme E is $\mathrm{Spec}\,\big(\mathrm{Sym}(G^\vee)\big)$, where $G_\bullet = \mathrm{Hom}_B(G, B)$. A section $X \to E$ is given by a map of B-algebras $\mathrm{Sym}(G^\vee) \to B$, which is completely determined by its restriction to the forms of degree one. This restriction is simply an element of $\mathrm{Hom}(G^\vee, B) \cong G^{\vee\vee} \cong G$, so that, up to isomorphism, G is recovered from the sections.

If $\mathcal{E}$ is a locally free sheaf of rank r on X and we have a morphism $f : Y \to X$ then $f^*\mathcal{E}$ is a locally free sheaf of rank r on Y. This is simply a consequence of the fact that if C is a B-algebra, $C \otimes_B B^r \cong C^r$ is free of rank r. When Y is a closed, open, or locally closed subscheme of X, $f^*\mathcal{E}$ is called the *restriction* of $\mathcal{E}$ to Y.

When $r = 1$, the vector bundle is called a *line bundle*.

We next want to characterize the first Chern class of a line bundle, and describe it explicitly in some important special cases.

Given a line bundle L on X, we can characterize the action of the first Chern class $c_1(L)$ by explaining, given V, a reduced and irreducible subscheme of dimension k, how one gets a class in $A_{k-1}(X)$: this class will be the value of $c_1(L) \cap [V]$ with $[V] \in \mathcal{A}_k(X)$ (this theory is also valid for $A_*(X)$). In fact we construct an element of $\mathcal{A}_{k-1}(V)$ and then use proper push-forward to get an element of $\mathcal{A}_{k-1}(X)$.

The first step is simply to restrict L to V. This gives a line bundle on V, and we need only see how to get a class in $\mathcal{A}_{k-1}(V)$ from a line bundle on V. Thus, for the purpose of this description, we might as well assume that $X = V$ and that L is simply a line bundle on V.

There is one case in which it is particularly easy to see how one gets a class from L: that is the case where L is associated with an effective Cartier divisor $D \subseteq X$. An *effective Cartier divisor* D on X is a closed subscheme of X that is defined, locally, by the vanishing of one element: in general, the element is required to be a nonzerodivisor, but in the case of V, which is reduced and irreducible, this simply means that the element is nonzero. An equivalent statement is that the sheaf of ideals defining D is locally free of rank one. This sheaf of ideals is therefore the sheaf of sections of a line bundle, which is denoted $\mathcal{O}_X(-D)$: its inverse (or dual) is denoted $\mathcal{O}_X(D)$.

When $L \cong \mathcal{O}_V(D)$, $c_1(L) \cap [V] = [D] \in \mathcal{A}_{k-1}(V)$: this makes sense, because D is a closed subscheme of V of pure dimension $k-1$. Note that L may be isomorphic with $\mathcal{O}_V(D)$ for more than one choice of D, but it is easy to show that in this situation, $[D] \in \mathcal{A}_{k-1}(V)$ is independent of the choice of D. When D is an effective Cartier divisor on X, if D does not contain V, and $L = \mathcal{O}_X(D)$, $c_1(L) \cap V = [D \cap V]$.

We next recall the notion of blowing up a Noetherian scheme X along a closed subscheme Z. First consider the case where X is $\mathrm{Spec}\,(B)$ and $Z = \mathrm{Spec}\,(B/I)$. Let $I = (f_0, \ldots, f_t)B$. Let $C_i \subseteq B_{f_i}$ be the ring which is generated over the image of B

by the images of the elements f_j/f_i, $0 \le j \le t$. Then the $\mathrm{Spec}\,(C_i)$ are open affines covering the blow-up $\mathrm{Bl}_Z X$ in this case. Note that $(C_i)_{f_j}$ and $(C_j)_{f_i}$ are isomorphic: both inject into $B_{f_i f_j}$, and the images are the same, which explains how the open affines paste. The resulting scheme is independent of the choice of generators of I, and may also be viewed as $\mathrm{Proj}_B B[IT]$, where $B[IT]$ is an $\mathbb{N}$-graded B-algebra, the Rees ring. Specifically,

$$B[IT] = B + IT + I^2T^2 + \cdots + I^nT^n + \cdots \subseteq B[T].$$

In the general case, X is covered by open affines $U_j = \mathrm{Spec}\,(B_j)$ on each of which $Z_j = Z \cap B_j$ is defined by an ideal $I_j \subseteq B_j$, and the schemes $\mathrm{Bl}_{Z_j} U_j$ can be pasted together in an obvious way to give $Bl_Z X$, because blowing up in the affine case commutes with localization. The scheme $\mathrm{Bl}_Z X$ is independent of the choice of the cover of X by open affines. There is a map $\mathrm{Bl}_Z X \to X$ which is a projective morphism and, in particular, a proper morphism.

Note that when X is reduced and irreducible, it is obvious that $\mathrm{Bl}_Z X$ and X have the same function field. Also observe that the map $\mathrm{Bl}_Z X \to X$ has the following very important property: the scheme-theoretic inverse image of Z is an effective Cartier divisor. To see this, it suffices to consider the affine case. The assertion then follows once one sees that if $I = (f_0, \ldots, f_t)B$ as above, then IC_i, which defines the intersection of the scheme-theoretic inverse of Z with $\mathrm{Spec}\,(C_i)$, is a principal ideal generated by a nonzerodivisor. But $IC_i = f_iC_i$, since $f_j = (f_j/f_i)f_i$ for all j, and f_i is a nonzerodivisor in $C_i \subseteq B_{f_i}$.

We can characterize the action of $c_1(L)$ in the general case as follows. As before, we may restrict L to V, where V is closed, reduced, and irreducible. Then there exists a proper morphism $f : Y \to V$, obtained by blowing up a suitably chosen closed subscheme of V, such that f^*L has the form $L_1 \otimes L_2^{-1}$, where L_1 and L_2 are associated with effective Cartier divisors. The L_i give classes β_i in $\mathcal{A}_{k-1}(Y)$. The class we seek is $f_*(\beta_1) - f_*(\beta_2)$.

Thus, line bundles L give rise to degree -1 operators $c_1(L) : \mathcal{A}_*(Y) \to \mathcal{A}_*(Y)$.

It turns out that all the operators $c_1(L)$ defined in this way commute.

We next note that given a vector bundle E of rank r on X, it has a projectivization $\mathbb{P}(E)$, and that the morphism $\mathbb{P}(E) \to X$ is both proper and also flat of relative dimension $r - 1$. If we cover X by open affines U such that $E|_U \to U$ is isomorphic to $\mathbb{A}^r_U \to U$, and consider the various transition functions used to glue these, we can use the same transition functions to clue the corresponding $\mathbb{P}_U \to U$: this gives the projectivization of E.

We can now characterize the Chern classes of a vector bundle of rank r: one can take a morphism $\pi : Z \to Y$ that is a composition of structural homomorphisms of projectivized vector bundles (these maps are flat and proper) such that the pullback of E has a filtration by line bundles. Then $c_i(E)$ is determined by the $i\,th$ elementary symmetric function of the first Chern classes of these line bundles, and lowers degrees by i. In this situation $\pi_* : \mathcal{A}_*(Z) \to \mathcal{A}_*(Y)$ is onto and π_* is a split monomorphism. Then $c_i(E) \cap \alpha = \pi_*(c_i(\pi^*(E) \cap \beta)$, where β is chosen so that $\pi_*(\beta) = \alpha$. These operators turn out to have values independent of the choices made.

These Chern classes are compatible with proper morphisms. If $Z \to Y$ is proper, E is a vector bundle on Y, and $\alpha \in A_*(Z)$, one has the *projection* formula

$$f_*\Big(c_i\big(f^*(E)\big) \cap \alpha\Big) = c_i(E) \cap f_*(\alpha).$$

All these operators commute and lower degree. One can therefore define new operators not only by substituting them in polynomials with rational coefficients, but also by substituting them in power series. All terms of sufficiently high order will lower degree so much that they may be interpreted as 0. In particular, for any line bundle L, we have an operator $e^{c_1(L)}$.

We can now define the Chern character of a vector bundle: this is the sum of the formal exponentials of the Chern classes of line bundles in a filtration. Here, one really needs rational coefficients. When there is no filtration one can still use the same method that we used for Chern classes to characterize these: pull back via a flat proper morphism to a scheme such that the pull-back does have a filtration by line bundles.

Similarly, we can define Todd classes: let $h(x)$ denote the formal power series for

$$x/(1-e^{-x}) = x/(x - x^2/2! + x^3/3! - \cdots) = 1/(1 - x/2 + x^2/6 - \cdots) = 1 + x/2 + \cdots .$$

In the case where E has a filtration by line bundles L_i we can take the product of $h\big(c_1(L_i)\big)$ to obtain the Todd class of E. We get a characterization of Todd classes in the general case by pulling back as above via a flat proper morphism to a scheme where there is a filtration by line bundles.