Commutative Algebra Seminar: Lecture of February 9, 2006

Finally, if $E_\bullet$ is a bounded complex of vector bundles on Y that is exact off a closed subscheme Z, and $\alpha \in \mathcal{A}_*(Y)$, one can define $\mathrm{ch}_Z^Y(E_\bullet) \cap \alpha)$ in $\mathcal{A}_*(Z)$: this pushes forward to $\sum_i (-1)^i \mathrm{ch}(E_i)$ in $\mathcal{A}_*(Y)$. These are compatible with proper morphisms and open immersions, have additivity and multiplicativity, and if D is an effective Cartier divisor on Y and $E_\bullet$ is the complex $0 \to \mathcal{O}_Y(-D) \to \mathcal{O}_Y \to 0$ ($\mathcal{O}_Y(-D)$ corresponds to the sheaf of ideals that defines D), then $\mathrm{ch}_D^Y(E_\bullet) \cap [Y] = \tau(N)^{-1} \cap [D]$, where N is the normal bundle to D in Y, which is the same as $\mathcal{O}_Y(D)|_D$, where $\mathcal{O}_Y(D) = \mathcal{O}_Y(-D)^{-1}$. For details we refer to Chapter 18 of [W. Fulton, *Intersection Theory*, Springer-Verlag, Berlin, 1984]. The proof of the existence of these local (or localized) Chern characters is based on a rather complicated graph construction due to MacPherson.

The properties listed in the paragraph above uniquely determine these local Chern characters.

We can now return to the situation we really want to study, in which we are working with $\mathcal{A}_*(X)$ where $X = \mathrm{Spec}\,(A)$ and A is a local ring that is a homomorphic image of a regular local ring. In developing these results, we will not need to refer to the theory of schemes.

Recall the local Riemann-Roch formula that we discussed in the Lecture of January 19.

(v) One has a local Riemann-Roch formula. Let z be the closed point of X. Then $\mathcal{A}_*(z) = \mathcal{A}_0(z) = \mathbb{Q}z$, which we identify with $\mathbb{Q}$, mapping $z \mapsto 1$. If $F_\bullet$ has homology of finite length, then for every finitely generated A-module M,

$$\chi(F_\bullet \otimes_A M) = \mathrm{ch}(F_*)\big(\tau(M)\big).$$

In this context it is worth remarking that one can understand the component $\tau_k(M)$ in degree $k = \dim(M)$: it is a linear combination of classes $[A/P]$ where P is minimal in the support of M such that $\dim\,(A/P) = k$. For such a P, the coefficient is the length of M_P over A_P.

In the case of (v) where $M = A$, we find that

$$\chi(F_\bullet) = \mathrm{ch}(F_*)\big(\tau(M)\big) = \sum_i \mathrm{ch}_i(F_\bullet)\big(\tau_i(A)\big).$$

Let $d = \dim\,(A)$. It will be crucial in the sequal that the term $\mathrm{ch}_d(F_\bullet)\big(\tau_d(A)\big)$ exhibits behavior that is, in some ways, better than the behavior of the Euler characteristic.

We now embark on the proof of the new intersection theorem in mixed characteristic p. We therefore assume that we have a counterexample $G_\bullet$ over a local ring A. Then $G_\bullet$ is a free complex with finite length homology such $H_0(G_\bullet) \neq 0$, and the length of the complex is smaller than $\dim\,(A)$. We may complete A and then kill a suitable minimal prime so as to obtain a counterexample where (A, m, K) is a complete local domain. We may assume that this counterexample has mixed characteristic p, since the result is known when the ring contains a field. Then A is module-finite over a formal power series ring $B = V[[x_2, \dots, x_d]]$ over a coefficient ring V that is a complete DVR with maximal ideal pV. Then V has a complete flat local extension V' with perfect (even algebraically closed) residue class field. Let $B' = V'[[x_2, \dots, x_d]]$, which is faithfully flat over B. We replace A by the faithfully flat extension $A' = A \otimes_B B'$, which is faithfully flat over A'. The maximal ideal m_B of B is clearly in the Jacobson radical of this ring. If we kill it, we obtain $A/m_B A \otimes_K L$ where m is nilpotent on $m_B A$: killing nilpotents, we get $K \otimes_K L \cong L$. Thus, A' is local, and of the same dimension as A, and mA' is primary to the maximal ideal of A'. We may therefore tensor $G_\bullet$ with A' to obtain a new counterexample over A'. We may have lost the domain property, but we may again kill a suitable minimal prime to recover it. Henceforth, we may assume that A is a complete local domain with perfect residue class field K.

Note that once we apply $\_ \otimes_A /pA$, we are in characteristic p. Therefore, we may assume the length of the complex is exactly $d-1$. Let $\overline{G}_\bullet = G_\bullet \otimes_A (A/pA)$.

The first key point is that we have a commutative diagram:

$$\begin{array}{ccc} \mathcal{A}_d\big(\mathrm{Spec}\,(A)\big) & \xrightarrow{\mathrm{ch}_{d-1}(G_\bullet)} & \mathcal{A}_1(x) \\ \big\downarrow & & \big\downarrow \\ \mathcal{A}_{d-1}\big(\mathrm{Spec}\,(A/pA)\big) & \xrightarrow{\mathrm{ch}_{d-1}(\overline{G}_\bullet)} & \mathcal{A}_0(x) \end{array}.$$

The vertical arrows are defined by intersecting with the effective Cartier divisor define by pA, or, equivalently, applying the first Chern class of the corresponding bundle. Starting in the group to the left in the top row, we may find the image of $[A]$ in two ways. Since the right hand group in the top row is 0, the image on the lower right is 0. This tells us that $\mathrm{ch}_{d-1}(\overline{G}_\bullet)([A/pA])$ is 0 in $A_0(x) \cong \mathbb{Q}$.

We shall complete the proof by showing that $\mathrm{ch}_{d-1}(\overline{G}_\bullet)([A/pA])$ is positive!

In his thesis, Sankar Dutta introduced the notion of the asymptotic Euler characteristic of a free complex with finite length homology over a local ring of prime characteristic $p > 0$ of dimension n. (In our case, $n = d-1$.) We want to discuss asymptotic Euler characteristics and their connection with local Chern characters. The definition is quite simple: if $G_\bullet$ is a free complex with finite length homology over a local ring R (in our case, $R = A/pA$, and what we are now calling $G_\bullet$ corresponds to what we previously called $\overline{G}_\bullet$) of prime characteristic $p > 0$, we let

$$\chi_\infty(G) = \lim_{e \to \infty} \frac{\chi\big(F^e(G_\bullet)\big)}{p^{ne}}.$$

This limit can be shown to exist by elementary means.

We can now complete the proof of the new intersection theorem by proving two facts.

Theorem. *With R and $G_\bullet$ as above,* $\mathrm{ch}_n(G_\bullet)([R]) = \chi_\infty(G_\bullet)$.

This is proved with the help of the local Riemann-Roch theorem.

Theorem. *With R and $G_\bullet$ as above,* $\chi_\infty(G_\bullet) > 0$.

Applying these results to the situation we obtained earlier with $R = A/pA$, we have the contradiction we need to prove the new intersection theorem! The second theorem is purely a result in commutative algebra, but is still quite non-trivial.

Proof of the first theorem. The Euler characteristic of the complex pulled back via the e th iterate F^e of the Frobenius endomorphism is the sum of the i th Chern characters of the pullback complex acting on $\tau_i([R])$. Pushing forward via Frobenius does not change this number in degree 0. But now it can be computed by letting the $\mathrm{ch}_i(G_\bullet)$ act on $\tau_i(R) \in \mathcal{A}_i\big(\mathrm{Spec}\,(R)\big)$. The action of Frobenius in degree i is multiplication by p^{ei} — we shall prove this in the sequel. Assuming this fact for the moment, and letting g denote the finite morphism $\mathrm{Spec}\,(R) \to \mathrm{Spec}\,(R)$ induced by F^e, we have

$$\chi\big(F^e(G_\bullet)\big) = \chi\big(g^*G_\bullet\big) = \sum_{i=0}^{n} \mathrm{ch}_i(g^*G_\bullet)\big(\tau_i([R])\big) =$$

$$g_*\Big(\sum_{i=0}^{n} \mathrm{ch}_i(g^*G_\bullet)\big(\tau_i([R])\big)\Big) = \sum_{i=0}^{n} \mathrm{ch}_i(G_\bullet) g_*\big(\tau_i([R])\big) = \sum_{i=0}^{n} p^{ie}\mathrm{ch}_i(G_\bullet)\big(\tau_i([R])\big).$$

We divide both sides by p^{ne} and take the limit as $e \to \infty$. All the terms in the sum corresponding to values of $i < n$ approach 0. This shows that $\chi_\infty(G_\bullet)$ exists, and is equal to $\mathrm{ch}_h(G_\bullet)\big(\tau_n([R])\big)$, and since $\tau_n([R]) = [R]$, our claim is established. $\square$