Commutative Algebra Seminar: Lecture of February 16, 2006

Our next objective is to analyze the action of the Frobenius endomorphism F on $\mathcal{A}_k(A)$ when (A, m, K) is a complete local ring of prime characteristic $p > 0$ and K is perfect. The key point is that this action is multiplication by p^k. It follows at once that the action of F^e is multiplication by p^{ke}. A key point is that the map $F : A \to A$ makes A into a module-finite A-algebra, i.e., that A is module-finite over $A^p = F(A)$.

In fact, if A has dimension n, it is module-finite over $B = K[[x_1, \ldots, x_n]]$, a formal power series ring over K, and B is free of rank p^n over $B^p = K[[x_1^p, \ldots, x_n^p]]$: the monomials $x_1^{a_1} \cdots x_n^{a_n}$ such that for all i, $0 \le a_i \le p-1$, form a free basis. It follows that A is module-finite over B^p, and then it is clear that A is module-finite over A^p, since $B^p \to A$ factors $B^p \to A^p \to A$. To complete the proof, we need only show that if $R = A/P$ is a domain of Krull dimension k, then the degree of the field extension $[\mathrm{frac}\,(R) : \mathrm{frac}\,(R^p)] = p^k$. After a change of notation, we may assume that $R = A$ is a domain of dimension n, and take B as before, so that B is free of rank p^n over B^p.

Let $\mathcal{K}$ be the fraction field of B and $\mathcal{L}$ the fraction field of A. Let $s = [\mathcal{L} : \mathcal{K}]$. Then $[\mathcal{L}^p : \mathcal{K}^p] = s$ as well, because the extensions $\mathcal{K} \to \mathcal{L}$ and $\mathcal{K}^p \to \mathcal{L}^p$ are isomorphic. In fact we have a commutative diagram

$$\begin{array}{ccc} \mathcal{L} & \xrightarrow{\cong} & \mathcal{L}^p \\ \uparrow & & \uparrow \\ \mathcal{K} & \xrightarrow{\cong} & \mathcal{K}^p \end{array}$$

where each horizontal isomorphism is induced by restricting the target of the Frobenius map $\gamma \mapsto \gamma^p$, and each vertical map is an inclusion. But then

$$[\mathcal{L} : \mathcal{K}^p] = [\mathcal{L} : \mathcal{K}][\mathcal{K} : \mathcal{K}^p] = sp^n,$$

since B is free of rank p^n over B^p. But

$$[\mathcal{L} : \mathcal{K}^p] = [\mathcal{L} : \mathcal{L}^p][\mathcal{L}^p : \mathcal{K}^p] = [\mathcal{L} : \mathcal{L}^p]s,$$

from which $[\mathcal{L} : \mathcal{L}^p] = p^n$ as well, as required. $\square$

We next want to prove that the asymptotic Euler characteristic of a free complex $G_\bullet$ over A of length $n = \dim(A)$ with finite length homology such that $H_0(G_\bullet) \neq 0$ is positive. This will occupy us for quite a while.

We first want to discuss the existence of a dualizing complex $D^\bullet$ for A. This will be a finite complex of injective modules $0 \to D^0 \to D^1 \to \cdots \to D^n \to 0$ such that D_i is the direct sum of the injective hulls of all the primes A/P such that P has dimension $n - i$, and whose homology consists of A-modules that are finitely generated. In our case, A is the quotient of a complete local domain of mixed characteristic p by a principal ideal (the ideal generated by p), and so is equidimensional, as well as a quotient of a regular local ring. For simplicity, we assume that A is equidimensional in the sequel.

We can construct $D^\bullet$ as follows. Map a regular ring $T \twoheadrightarrow A$, so that $A = T/\mathfrak{A}$ for a certain ideal $\mathfrak{A}$ of T. Let T have dimension N. We write $E_S(M)$ for the injective hull of the S-module M over the ring S. Let $E^\bullet$ be a minimal injective resolution of T. We know that the Bass numbers of a prime Q of T are the vector space dimensions of the modules $\mathrm{Ext}^j_{T_Q}(T_Q/QT_Q,\, T_Q)$ over the field T_Q/QT_Q: for each Q, these are all 0 except when j is the height of Q, and then the value is one. Thus, E^i is the direct sum over T of all the injective hulls of the T/Q such that the dimension of T/Q is $N - i$, i.e., such that the height of Q is i. It follows that $E^\bullet$ is a dualizing complex for T. Now consider the complex $\mathrm{Hom}_T(A,\, E^\bullet)$. If $\mathfrak{A} \subseteq Q$, so that Q corresponds to a prime $\overline{Q} = Q/\mathfrak{A}$ of A, then $\mathrm{Hom}_T\bigl(A,\, E_T(T/Q)\bigr)$ may be identified with $E_A(A/\overline{Q})$. If Q does not contain $\mathfrak{A}$, then $\mathrm{Hom}_T\bigl(A,\, E_T(T/Q)\bigr) = 0$: no nonzero element of $E_T(T/Q)$ can be killed by $\mathfrak{A}$, for then some associated prime of $E_T(T/Q)$ would contain $\mathfrak{A}$, and the only associated prime is Q.

The cohomology of the complex $\mathrm{Hom}_T(A,\, E^\bullet)$ consists of the modules $\mathrm{Ext}^\bullet_T(A, T)$. These are finitely generated T-modules killed by $\mathfrak{A}$, and so are finitely generated A-modules, as required. Notice that the complex $\mathrm{Hom}_T(A,\, E^\bullet)$ is 0 in degree $<$ height $(\mathfrak{A}) = h$. We may now shift degrees by h to obtain the dualizing complex $D^\bullet$: $D^i = \mathrm{Hom}_T(A,\, E^{i-h})$.

From the description of $D^\bullet$, its j th cohomology module, which we denote M_j, has dimension at most $n - j$: it will be a homomorphic image of a submodule of D^j generated by finitely many elements (these can be chosen to map onto its generators), and this module must have dimension at most $n - j$, since all of its associated primes have that dimension.

Now consider the double complex $\mathrm{Hom}_R\bigl(F^e(G_\bullet),\, D^\bullet\bigr)$:

$$
\begin{array}{ccccccccc}
 & & 0 & & & & 0 & & \\
 & & \downarrow & & & & \downarrow & & \\
0 & \to & \mathrm{Hom}\bigl(F^e(G_0),\, D^0\bigr) & \to & \cdots & \to & \mathrm{Hom}\bigl(F^e(G_n),\, D^0\bigr) & \to & 0 \\
 & & \downarrow & & & & \downarrow & & \\
 & & \vdots & & & & \vdots & & \\
 & & \downarrow & & & & \downarrow & & \\
0 & \to & \mathrm{Hom}\bigl(F^e(G_0),\, D^n\bigr) & \to & \cdots & \to & \mathrm{Hom}\bigl(F^e(G_n),\, D^n\bigr) & \to & 0 \\
 & & \downarrow & & & & \downarrow & & \\
 & & 0 & & & & 0 & &
\end{array}
$$

We consider the two spectral sequences for this double complex. First note that the rows are exact except for the bottom row. To see this, note that D^i for $i < n$ is a direct sum of terms $E = E_A(A/P)$ each of which is a module over A_P for P strictly contained in m. For such an E, $\mathrm{Hom}_A\bigl(F^e(G_\bullet),\, E\bigr) \cong \mathrm{Hom}_{A_P}\bigl(F^e(G_\bullet)_P,\, E\bigr)$. But $F^e(G_\bullet)_P$ is an exact sequence of free modules over A_P, and so breaks up into split short exact sequences, and it follows that applying $\mathrm{Hom}_{A_P}(\_\,,\, E)$ preserves exactness.

Now consider the bottom row. From our description of $D^\bullet$, we know that $D^n = E_A(K)$, the injective hull of the residue class field. Thus, the homology of the bottom row is the

Matlis dual of the homology of $F^e(G_\bullet)$: each module that occurs has the same length as the corresponding homology of module of $F^e(G_\bullet)$.

It follows that we can say the same for the homology of the total complex, and so we may use the homology of the total complex to study the asymptotic Euler characteristic.

We next study what happens when we take homology first of columns and the of rows. After we take homology of columns, the homology of a typical row is the homology of $\mathrm{Hom}\big(F^e(G_\bullet),\, M_j\big)$ where $\dim(M_j) \le n-j$. We shall show this is bounded by $C_j p^{e(n-j)}$ for a certain positive real constant C_j: see the first part of the Lemma below. The terms that contribute to $\mathrm{Hom}\Big(H_i\big(F^e(G_\bullet)\big), E_A(K)\Big)$ for $i > 0$ come from (k, j) terms with $k + j = i + n$, and we know that $k \le n$ (this is where we are using that $G_\bullet$ has at most n terms). But then we have that $j \ge i > 0$. There are only finitely many of these terms, and each makes a contribution bounded by a positive real constant multiplied by $p^{e(n-1)}$. These terms therefore all contribute 0 to the asymptotic Euler characteristic, and we can conclude that the asympotic Euler characteristic is the same as

$$\lim_{e\to\infty} \frac{\ell\Big(H_0\big(F^e(G_\bullet)\big)\Big)}{p^{en}}.$$

The second part of the Lemma stated below shows that this limit is positive.

Lemma. *Let $(A,\, m,\, K)$ be a local ring of prime characteristic $p > 0$. Let $G_\bullet$ be a finite free complex of finitely generated free modules whose homology has finite length and such that $H_0(G_\bullet) \ne 0$.*

(a) *If M is a finitely generated R-module of dimension k, then there is a positive real constant C such that for all e,*

$$\ell\Big(H_i\big(\mathrm{Hom}_A\big(F^e(G_\bullet),\, M\big)\big)\Big) \le Cp^{ke}.$$

(b) *The quantity*

$$\frac{\ell\Big(H_0\big(F^e(G_\bullet)\big)\Big)}{p^{en}}$$

is bounded away from 0 as $e \to \infty$.

Once we have proved this Lemma, the proof of the New Intersection Theorem in mixed characteristic $p > 0$ will be complete.