Math 711: Lecture of September 7, 2005

Our objective is to present a large number of open questions in commutative algebra, to give an indication of the status of these questions, as well as to develop many techniques of attack that may prove useful in addressing these questions.

At the same time, we shall also present some of the consequences of affirmative answers to these questions, proving the needed implications. The extent to which these proofs will be given here depends somewhat on the accessibility of the proofs in other references in the literature.

All rings are assumed to be commutative, associative, with identity, and all modules unital, unless otherwise specified.

As an example of a very simple open question, let $X = (x_{ij})$ and $Y = (y_{ij})$ be matrices of $2n^2$ algebraically independent indeterminates over a field K. Let $I_t(A)$ denote the ideal generated by the size t minors of the matrix A (the ring is deduced from the context). Let K be a field, let $K[X, Y]$ be the polynomial ring in the entries of X and Y over K (we shall typically use this notation when dealing with one or more matrices) and let $R = K[X, Y]/I_1(XY - YX)$.

It is an open question, in general, whether R is reduced, a domain, normal, and/or Cohen-Macaulay. These properties all hold when the matrices have size 3 or smaller.

We shall discuss methods that might be used to attack this problem, and use them to prove that, with K as above, for an $r \times s$ matrix of indeterminates X, one has that $K[X]/I_t(X)$ is a Cohen-Macaulay normal domain. The method we shall focus on is that of principal radical systems. Another approach that is sometimes useful is the theory of algebras with straightening law: if time permits, we will discuss that theory as well.

In both questions, one can reduce at once to the case where K is algebraically closed. Note that in R above, the images of X and Y are "generic" commuting matrices, and the algebraic set $V(I_1(XY - YX))$ consists of pairs of commuting $n \times n$ matrices, a natural object of study. This algebraic set is known to be a variety, i.e., to be irreducible, by a result of M. Gerstenhaber.

In quite a different direction, we shall consider several homological conjectures that are stated in or reduce to the case of a local ring, and where it, in fact, suffices to consider the case of a complete local ring.

One is the existence of big and small Cohen-Macaulay modules and algebras. The existence of big Cohen-Macaulay algebras and the existence of small Cohen-Macaulay modules both imply the existence of big Cohen-Macaulay modules. Big Cohen-Macaulay modules and algebras are known to exist in equal characteristic, and in mixed characteristic in dimension at most 3. Small Cohen-Macaulay modules are not known to exist even in dimension 3, and the conjecture that they do exist at this point seems doubtful to me.

Another is the direct summand conjecture, which asserts that a regular Noetherian ring is a direct summand of every module-finite extension ring. We shall prove the equivalence of this conjecture with several others, including the *canonical element conjecture*, the *monomial conjecture*, and with the improved form of the new intersection theorem, and we shall show it can be used to deduce a conjecture of M. Auslander, an affirmative answer to a question of Bass, and a proof of the *syzygy conjecture* of Evans and Griffith. The conjecture of Auslander is that a zerodivisor on a finitely generated module $M \neq 0$ of finite projective dimension over a local ring (R, m, K) must be a zerodivisor on R. Contrapositively, a nonzerodivisor (or, by a trivial induction, even a regular sequence) in m on such a module M must be a nonzerodivisor (respectively, a regular sequence) on R. Bass's question is whether, if a local ring (R, m, K) possesses a finitely generated module $M \neq 0$ of finite injective dimension, must R be Cohen-Macaulay. We shall show how these follow from new intersection theorem, which is explained below, and hence how they follow from the direct summand conjecture.

The new intersection theorem and hence the two corollaries of it described above have been proved even in mixed characteristic by Paul Roberts using the Fulton-MacPherson-Baum intersection theory. We shall also explain this proof, but we assume certain facts about intersection theory without proof.

The Evans-Griffith syzygy conjecture asserts, in its simplest form, that any k th module of syzygies over a regular local ring, if not free, needs at least k generators. The hypothesis on the ring can be weakened. The problem is still open in mixed characteristic, in the same cases where the direct summand conjecture is open. It can be deduced from the improved form of the new intersection theorem, which is equivalent to the direct summand conjecture, and we shall eventually explain that argument.

We want to explain in greater detail what the three conjectures (monomial, canonical element, improved new intersection) say, as well as the new intersection theorem itself.

The monomial conjecture asserts that if $x_1, \ldots, x_d$ is a system of parameters for a local ring R of dimension d, then for every positive integer t, one has that

$$x_1^t \cdots x_d^t \notin (x_1^{t+1}, \ldots, x_d^{t+1})R.$$

Both the new intersection and its improved version refer to a finite free complex $G_\bullet$ of finite rank free modules, say

$$0 \to G_n \to \cdots \to G_1 \to G_0 \to 0,$$

over a local ring (R, m, K). Both conjectures assume that the homology modules $H_i(G_\bullet)$ have finite length for $i \geq 1$, and both assume that the augmentation module $H_0(G_\bullet) \neq 0$. In the intersection theorem, it is assumed that $H_0(G_\bullet)$ has finite length. In the improved version the hypothesis is weakened: it is only assumed that some minimal generator of $H_0(G_\bullet)$ is killed by a power of m. In both cases, the conclusion is that $\dim(R) \leq n$. The hypothesis in the improved version is more technical, and the resulting conjecture may not seem a lot stronger at first, but it turns out to be, apparently, much stronger in its applications. The new intersection theorem implies both Auslander's zerodivisor conjecture and an affirmative answer to Bass's question, and it also proves the Evans-Griffith syzygy conjecture. All of these conjectures are known if the ring contains a field, but while the new intersection theorem has been proved by Paul Roberts in all cases, the improved version, like the direct summand conjecture, remains open in mixed characteristic in dimension 4 or more.

The canonical element conjecture is a bit harder to explain. One version is this: Let (R, m, K) be local of Krull dimension d, choose a free resolution of K over R by finitely generated free modules, truncate this resolution to obtain an exact complex $\mathcal{C}_\bullet$ of the form

$$0 \to \mathrm{syz}^d(K) \to R^{b_{d-1}} \to \cdots \to R^{b_1} \to R^{b_0} \to 0,$$

choose a system of parameters $x_1, \ldots, x_d$ for R, and choose a lifting of the obvious map of augmentations $R/(x_1, \ldots, x_d) \twoheadrightarrow R/m = K$ to a map from the Koszul complex $\mathcal{K}_\bullet(x_1, \ldots, x_d;\, R)$ to the complex $\mathcal{C}_\bullet$. Then, the conjecture asserts that no matter how the choices are made, the map $R = K_d(x_1, \ldots, x_d; R) \to \mathrm{syz}^d(K)$ is not zero. Equivalently, the element in $\mathrm{syz}^d(K)$ that is the image of $1 \in R$ is not zero.

Because the map of complexes is not unique, the image of 1 in $M = \mathrm{syz}^d(K)$ is only determined mod $(x_1, \ldots, x_d)M$. Therefore, an equivalent assertion is that the image of 1 in $M/(x_1, \ldots, x_d)M$, where $M = \mathrm{syz}^d(K)$, is not 0. Now consider what happens when we replace the system of parameters $x_1, \ldots, x_d$ by the system of parameters $x_1^t, \ldots, x_d^t$. There is a map of Koszul complexes

$$\mathcal{K}_\bullet(x_1^t, \ldots, x_d^t;\, R) \to \mathcal{K}_\bullet(x_1, \ldots, x_d;\, R)$$

that is the identity on R and whose matrix mapping $R^d \to R^d$ in degree one is the diagonal matrix with diagonal entries $x_1^{t-1}, \ldots, x_d^{t-1}$. If one thinks of the Koszul complex as constructed using exterior powers, the other maps are the exterior powers of the map in degree one, and so the last map $R \to R$ is given by multiplication by the determinant of the matrix, i.e., by multiplication by $x_1^{t-1} \cdots x_d^{t-1}$. By composing with the map already constructed from $\mathcal{K}_\bullet(x_1, \ldots, x_d;\, R) \to \mathcal{C}_\bullet$, we obtain a map from $\mathcal{K}_\bullet(x_1^t, \ldots, x_d^t;\, R) \to \mathcal{C}_\bullet$. The image of 1 is the product of the earlier image by $x_1^{t-1} \cdots x_d^{t-1}$.

The modules $M_t = M/(x_1^t, \ldots, x_d^t)M$ form a direct limit system in which the maps $M_t \to M_{t+1}$ are induced by multiplication by $x_1 \ldots x_d$ on M, and the direct limit is $H^d_{(x_1, \ldots, x_d)}(M) \cong H^d_m(M)$. We refer to the element ϵ that is the image of 1 from any of the maps $R = \mathcal{K}_d(x_1^t, \ldots, x_d^t;\, R) \to M$ in the local cohomology module $H^d_m(\mathrm{syz}^d(K))$ as the *canonical element* in $H^d_m(\mathrm{syz}^d(K))$. The canonical element conjecture asserts that $\epsilon \neq 0$.

If we take a different free resolution of K we get a different complex $\mathcal{C}'_\bullet$ and a different dth module of syzygies M'. But there are induced maps of complexes between $\mathcal{C}_\bullet$ and $\mathcal{C}'_\bullet$ in both directions whose compositions are, up to homotopy, the appropriate identity. It follows that there are induced maps $H^d_m(M) \to H^d_m(M')$ and $H^d_m(M') \to H^d_m(M)$, and these maps take the canonical element in $H^d_m(M)$ to the canonical element in $H^d_m(M')$ and vice versa. Therefore, whether the canonical element vanishes is independent of the choice of free resolution of K.

There is a more invariant point of view that identifies ϵ with an element of $\mathrm{Tor}^R_d(K,\, H^d_m(R))$: we shall discuss this later.

We shall also consider a strengthening of the direct summand conjecture (the strong direct summand conjecture) and its many consequences.

We shall likewise discuss the Buchsbaum-Eisenbud-Horrocks conjecture that the jth Betti number of a finite length module $M \neq 0$ over a regular local ring (R, m, K) is at least $\binom{n}{j}$, which is open in dimension 5 or more. The j th Betti number is the rank of the j th free module in a minimal free resolution.

Another intriguing open question is this, known as *Lech's conjecture*. If $(R, m, K) \to (S, n, L)$ is flat local, is the multiplicity of R bounded by the multiplicity of S? We review the definition of this multiplicity. Recall that the Hilbert function of a finitely generated module M over a local ring (R, m, K) is given by the formula $H_M(n) = \ell(M/m^n M)$, where ℓ denotes length. This agrees with a polynomial in n for all $n \gg 0$, and the degree of this polynomial is equal to the Krull dimension r of M (equivalently, of $R/(\mathrm{Ann}_R M)$). The leading term of this polynomial has the form $\dfrac{e}{r!}n^r$, where e is a positive integer. When $M = R$, or when $\dim(M) = \dim(R)$, so that $r = d$, we refer to $e = e(R)$ (or $e(M)$) as the *multiplicity* of R (or of M). Evidently, one can also recover $e(M)$ as

$$d! \lim_{n\to\infty} \frac{\ell(M/m^{n+1}M)}{n^d}.$$

Regular local rings have multiplicity one.

A finitely generated module M over a local ring R is called a *maximal* Cohen-Macaulay module if $\mathrm{depth}_m M = \dim(R)$. Thus, such an M is Cohen-Macaulay of largest possible dimension. We shall see that for such an R-module M, $\nu(M) \le e(M)$. When equality holds, M is called a *linear maximal Cohen-Macaulay module*. The term *maximally generated* Cohen-Macaulay module is also used.

One approach to Lech's conjecture in certain cases is to study linear maximal Cohen-Macaulay modules, and even a weakening of this notion, and we shall do this. Lech's conjecture is open even when one local ring is module-finite and free over the other.

It is also an open question whether every Cohen-Macaulay local ring has a linear maximal Cohen-Macaulay module.

We note that the notion of linear maximal Cohen-Macaulay module is also studied in the graded case.