**Math 711: Lecture of September 9, 2005**

We note that if $M$ is a linear maximal Cohen-Macaulay module over the local ring $(R, m, K)$ and $x \in m - m^2$ is in sufficiently general position (we shall make all this precise later, when we return to a more detailed study of this topic), then neither the multiplicity of $M$ nor the least number of generators of $M$ changes when we pass to $M/xM$ and $R/xR$.

In the dimension 0 case, the multiplicity of $M$ is its length, and so $\nu(M) \leq e(M)$. In the general case, the multiplicity of $M$ is the length of $M/(x_1, \ldots, x_d)M$ for a system of parameters in sufficiently general position (one may need to enlarge the residue field to show that such a system of parameters exists). In any case, $\nu(M) \leq e(M)$ in general.

We shall next focus, for a while, on the problem of proving that ideals in (usually) polynomial rings are radical and/or prime by working with large classes of such ideals. This is the method of *principal radical systems* discussed last time.

It is based on two simple lemmas.

**Lemma.** *Let $R$ be a Noetherian ring that is either local or $\mathbb{N}$-graded, and let $x \in R$ be in the maximal ideal or be a form of positive degree. Suppose that $N$ is the nilradical of $R$, that $N$ is prime, that $x \notin N$, and that $R/xR$ is reduced. Then $N = 0$, i.e., $R$ is a domain.*

*Proof.* Suppose that $u \in N$. Since $R/xR$ is reduced, we must have that $u = xv$ for some $v \in R$. Since $xv \in N$, $x \notin N$, and $N$ is prime, we must have that $v \in N$. Therefore $N = xN$. By Nakayama's lemma for local or graded rings, $N = 0$. $\square$

By applying this Lemma to $R/I$ in the situation below, we obtain:

**Corollary.** *Let $R$ be a Noetherian ring that is either local or $\mathbb{N}$-graded, and let $x \in R$ be in the maximal ideal or be a form of positive degree. Suppose that $I$ is a (homogeneous in the graded case) proper ideal of $R$ with radical $P$, where $P$ is prime, that $x \notin P$, and that $P + xR$ is radical. Then $I = P$, i.e., $I$ is prime.*

The next Lemma has various generalizations that may prove useful, but we shall stick with the simplest case.

**Lemma.** *Let $R$ be Noetherian, let $I$ be an ideal of $R$, let $J$ be the radical of $I$, and suppose that $J \subseteq P$ where $P$ is prime. Suppose that $I+xR$ is radical where $x \notin P$, and that $xP \subseteq I$. Then $I = J$ , i.e., $I$ is radical.*

*Proof.* Suppose that $u \in J$. Then $u \in I + xR$, say $u = i + xr$, where $i \in I$ and $r \in R$. Then $xr = u - i \in J \subseteq P$, and so $r \in P$. Since $xP \subseteq I$, we have that $xr \in I$ and so $u = i + xr \subseteq I$. $\square$

We want to use these lemmas to prove the following result:

**Theorem.** *Let $K$ be a field, let $r$ and $s$ be positive integers, let $t$ be an integer with $1 \leq t \leq \min\{r, s\}$, and let $X$ be an $r \times s$ matrix of indeterminates over $K$. Then $I_t(X)$ is a prime ideal, i.e., $K[X]/I_t(X)$ is a domain.*

The proof will take a while. The idea is to include $I_t(X)$ in a much larger, but finite, family of ideals to which we can apply the lemmas above. The ideals are typically radical rather than prime. The result is proved by reverse induction, in that the largest ideal(s) in the family are shown to be radical first. The family has the property that for each ideal $I$ in it that is not maximal in the family, there is a larger ideal of the form $I + xR$ in the family, which will be known to be radical from the induction hypothesis.

We shall show first that the ideals $I_t(X)$ have radicals that are prime. Thus, once we show that they are radical, it will follow that they are prime.

Note that if $L$ is the algebraic closure of $K$ and $R$ is a $K$-algebra, $R \subseteq L \otimes_K R$, and so to show that $R$ is reduced or a domain it suffices to show the corresponding fact for $L \otimes_K R$. Thus, the problem we are discussing reduces to the case where $K$ is algebraically closed, and we assume this from here on. This will enable us to take a naive approach to the material we need from algebraic geometry, which will involve only basic facts about closed algebraic sets in affine spaces $\mathbb{A}^N_K$.

Note that showing that $\mathrm{Rad}\,(I_t(X))$ is prime is equivalent to showing that $V(I_t(X))$ is an irreducible closed algebraic set. We think of points of $\mathbb{A}^{rs}_K$ as corresponding to $r \times s$ matrices over $K$. Then $V(I_t(X))$ is precisely the set of $r \times s$ matrices of rank $\leq t-1$.

**Proposition.** *Let $r$, $s$, and $t$ be as above. Let $A$ be an $r \times s$ matrix over a field $K$. Then $A$ has rank $\leq t-1$ if and only if $A$ factors $BC$ where $B$ is an $r \times (t-1)$ matrix over $K$ and $C$ is a $(t-1) \times s$ matrix over $K$.*

*Proof.* We think of $A$ as giving a linear map $K^s \to K^r$, where $K^s$ is interpreted as $s \times 1$ columns. The rank is at most $t-1$ if and only if the image has dimension $\leq t-1$, i.e., if and only if the map factors $K^s \to K^h \to K^r$ where $h \leq t-1$. We may think of $K^{t-1}$ as $K^h \oplus K^{t-1-h}$ and extend the map $K^h \to K^r$ to the additional summand $K^{t-1-h}$ by letting it be 0. This gives a factorization $K^s \to K^{t-1} \to K^r$ for $A$ which yields that $A = BC$, as required, while any linear map with such a factorization obviously has rank at most $t-1$. $\square$

**Corollary.** *With notation as above, $V(I_t(X))$ is irreducible.*

*Proof.* Think of $\mathbb{A}^{(r+s)(t-1)} \cong \mathbb{A}^{r(t-1)}_K \times \mathbb{A}^{(t-1)s}_K$ as indexing pairs of matrices $(B,\, C)$ where $B$ is $r \times (t-1)$ and $C$ is $(t-1) \times s$. We have a map $\mathbb{A}^{(r+s)(t-1)} \to V((I_t(X))$ that sends $(B,\, C) \mapsto BC$, and by the preceding Proposition this map is surjective. Since $\mathbb{A}^{(r+s)(t-1)}$ is irreducible and the image of an irreducible is irreducible, $V(I_t(X))$ is irreducible. $\square$

Of course, this establishes that $\mathrm{Rad}\,(I_t(X))$ is prime.

For heuristic reasons, we now carry through the proof that $I_t(X)$ is radical first for the case where $t = 2$. Let $J_{k,h,a}(X) = J_{k,h,a}$ denote the ideal generated by the entries of the first $h$ rows of $X$, the first $k$ columns of $X$, and the first $a$ entries of the $(h+1)$ st row of $X$. Here, $0 \leq k \leq s$, $0 \leq h \leq r$, and $0 \leq a \leq s$. If $h = r$ or $k = s$ all the variables have been killed and $a = 0$ is forced. We also abbreviate $J_{k,h,0} = J_{k,h}$ and $J_{0,0,a} = J_a$. Note that $J_{k,h,a} = J_{k,0} + J_{0,h,a}$. If $a \leq k$, $J_{k,h,a} = J_{k,h}$. Certain ideals have more than one description: e.g., if $h < r$, $J_{k,h,s} = J_{k,h+1,0}$.

We shall prove by induction that all of the ideals $I_2(X) + J_{k,h,a}(X)$ are radical, and prime if $a = 0$. We assume the result for smaller matrices of indeterminates. Evidently, $I_2(X) + J_{s,r} = J_{s,r}$ is the ideal generated by all the indeterminates and is maximal. We now consider one ideal $I = I_2(X) + J_{k,h,a}(X)$ in the family, and assume that all larger ideals in the family are radical. We need to show that $I$ is radical.

We can simplify things a bit as follows. Let $X'$ be the $(r-h) \times (s-k)$ matrix in the lower right corner of $X$. As noted above we may assume that $a \geq k$. Then we have an obvious isomorphism

$$K[X]/\big(I_2(X) + J_{k,h,a}(X)\big) \cong K[X']/\big(I_2(X') + J_{a-k}(X')\big)$$

induced by the $K$-algebra surjection $K[X] \twoheadrightarrow K[X']$ that fixes each indeterminate in $X'$ while sending the other indeterminates to 0. Since we know the result for the smaller matrix $X'$ if either $h$ or $k$ is positive, there is no loss of generality in assuming that $h = k = 0$. Likewise, we may assume that $0 \leq a \leq s-1$. Finally, if either $r$ or $s$ is 1, then $I_2(X) = 0$, and the ideal is generated by a subset of the variables and is clearly prime. Henceforth we assume that $r, s \geq 2$.

Thus, $I = I_2(X) + J_a$ where $0 \leq a \leq s-1$. Let $x = x_{1,a+1}$. Since we know that larger ideals in the family are radical, we have that $I_2(X) + J_{a+1}$ is radical, and this is $I + (x)$. We consider two cases.

(1) $a = 0$. In this case, we know that $\mathrm{Rad}\,(I)$ is prime. The result now follows from the corollary to the first lemma, provided that we know that $x$ is not in the radical of $I$. This follows because we can specialize $x_{11}$ to 1 and all other variables to 0 and we get a point of $V(I)$ where $x_{11} \neq 0$. $\square$

(2) $1 \leq a \leq s-1$. For every i,$j$ such that $2 \leq i \leq r$, $1 \leq j \leq a$, consider the $2 \times 2$ submatrix of $X$ formed by the intersection of the first and $i\,th$ rows of $X$ with the $j$ th and $a+1$ st columns, namely:

$$\begin{pmatrix} x_{1,j} & x_{1,a+1} \\ x_{i,j} & x_{i,a+1} \end{pmatrix}.$$

The determinant of this matrix is in $I_2(X)$, and so $x_{1,j}x_{i,a+1} - x_{i,j}x \in I_2(X) \subseteq I$. Since $x_{1,j} \in J_a \subseteq I$ as well, we have that $xx_{i,j} \in I$. Let $P = I_2(X) + J_{a,0}(X)$. This is a larger ideal of our family, and is therefore radical, by the induction hypothesis. But the quotient by it is $\cong K[X']/I_2(X')$, where $X'$ is the submatrix of $X$ formed by the last $s-a$ columns of $X$, and so the radical is prime. Thus, $P$ is a prime ideal containing $J$, and is generated over $I$ by the elements $x_{i,j}$, $2 \leq i \leq r$, $1 \leq j \leq a$. It follows that $xP \subseteq I$. Finally, $x \notin P$, since we get a point of $V(P)$ by specializing so that $x_{1,a+1} = 1$ while every other indeterminate is specialized to 0. The fact that $I$ is radical now follows from the second lemma. $\square$