Math 711: Lecture of September 12, 2005

We next want to use the work that we have done on the ideals $I_2(X)$ to prove that the rings $K[X]/I_2(X)$ are all Cohen-Macaulay rings. We first need to calculate the dimensions of the rings $K[X]/I_2(X)$.

Proposition. *Let X be an $r \times s$ matrix of indeterminates. The localization of $R = K[X]/I_2(X)$ at the element $x = x_{1,1}$ is isomorphic with the localization of the polynomial ring $S = K[x_{i,1}, x_{1,j} : 1 \le i \le r,\ 1 \le j \le s]$ at the element $x = x_{1,1}$. Hence,* $\dim(R) = \dim(R_x) = r + s - 1$.

Proof. For $i \ge 2$, $j \ge 2$ the equation given by the vanishing of the 2×2 minor formed the first and ith rows and the first and jth columns is

$$x_{1,1}x_{i,j} - x_{1,j}x_{i,1} = 0$$

which is equivalent to

$$(*) \quad x_{i,j} = x_{1,j}x_{i,1}/x$$

in R_x. Consider the K-algebra homomorphism $K[X] \to S_x$ that fixes $x_{i,j}$ if $i = 1$ or if $j = 1$, and otherwise sends $x_{i,j} \mapsto x_{1,j}x_{i,1}/x$. It is straightforward to verify that the map kills $I_2(X)$ and so induces a surjection $R_x \twoheadrightarrow S_x$. The inclusion $S \subseteq K[X]$ induces a map $S_x \to R_x$. The composition $(R_x \to S_x) \circ (S_x \to R_x)$ is clearly the identity on S_x, and the composition $(S_x \to R_x) \circ (R_x \to S_x)$ is the identity on R_x because of the displayed relations $(*)$. The statement about dimensions is clear, since $\dim(S_x) = \dim(S) = r + s - 1$. $\square$

We may also argue as follows in calculating the dimension of R, which is the same as the dimension of $V(I_2(X))$. Consider the open set where the first row of the matrix is nonzero. The first row varies in $\mathbb{A}^s_K$ (with the origin deleted), i.e., in a variety of dimension s. Each of the other $r-1$ rows is a scalar times the first row, and so one expects the dimension to be $s + (r-1) = r + s - 1$.

Theorem. *With r, s, X as above, each of the rings $K[X]/I_2(X)$ is Cohen-Macaulay.*

Before proving this, we note the following.

Lemma. *If I and J are any ideals of any ring R, there is an exact sequence:*

$$0 \to R/(I \cap J) \to R/I \oplus R/J \to R/(I+J) \to 0$$

where the first map sends $r + (I \cap J) \mapsto (r + I) \oplus (-r + J)$ and the second map sends $(r + I) \oplus (r' + J) \mapsto (r + r') + (I + J)$.

Proof. It is straightforward to check that the maps are well-defined and R-linear. The first map is injective, since $r + (I \cap J)$ is in the kernel iff $r \in I$ and $r \in J$. The second map obviously kills the kernel, and is clearly surjective, Finally, $(r + I) \oplus (r' + J)$ is in the kernel of the second map iff $r + r' \in I + J$, i.e., $r + r' = i + j$ with $i \in I$ and $j \in J$. But then $r - i = r' + j = r_0$, and $(r + I) \oplus (r' + J)$ is the image of $r_0 + (I \cap J)$. $\square$

In the case of an $\mathbb{N}$-graded Noetherian ring R with $R_0 = K$, a field, and homogeneous maximal ideal m, R is Cohen-Macaulay if and only if $\mathrm{depth}_m R = \dim(R)$.

Proof of the Theorem. We use induction, and so we may assume the result if either or both of the dimensions of the matrix X are decreased. R is a domain and $x = x_{1,1}$ is therefore a nonzerodivisor. It will therefore suffice to prove that R/xR is Cohen-Macaulay. In R/xR, the fact that $x_{i,1}x_{1,j} - x_{1,1}x_{i,j} = 0$ shows that any minimal prime of x either contains all the $x_{1,j}$ or all of the $x_{i,1}$. Let P be the ideal $I_2(X) + (x_{1,j} : 1 \le j \le s)$ and Q the ideal $I_2(X) + (x_{i,1} : 1 \le i \le r)$. It follows that $V(x) = V(P) \cup V(Q)$. Since all of these ideals are radical, we have that $xR = P \cap Q$.

Let X', X'', and X''' be the matrices obtained from X be deleting, respectively, the first row, the first column, and both the first column and row. Then $R/P \cong K[X']/I_2(X')$ is a Cohen-Macaulay domain of dimension $(r-1)+s-1 = r+s-2$ by the induction hypothesis. $R/Q \cong K[X'']/Q$ is, similarly, a Cohen-Macaulay domain of dimension $r + (s-1) - 1 = r + s - 2$. Moreover, $K[X]/(P+Q) \cong K[X''']/I_2(X''')$ is a Cohen-Macaulay domain of dimension $(r-1) + (s-1) - 1 = r + s - 3$, again using the induction hypothesis. We can now make of the short exact sequence

$$0 \to R/xR \to R/P \oplus R/Q \to R/(P+Q) \to 0.$$

Since the module in the middle has depth $r + s - 2$ on m and the module on the right has depth $r + s - 3$ on m, the module on the left has has depth $r + s - 2$ on m. (One may use the long exact sequence for $Ext_R(K, \_)$, or for Koszul homology, or for local cohomology to show this.) Since x is not a zerodivisor in the domain R, it follows that $\mathrm{depth}_m(R) = r + s - 1$, which is $\dim(R)$. Therefore, R is Cohen-Macaulay. $\square$

We now want to generalize all this to the case of $t \times t$ minors. We introduce two notations that will be useful in dealing with matrices. If A is a matrix, we write $A|_t$ for the submatrix formed from the first h columns of A. If A and B are matrices of sizes $r \times t$ and $r \times u$, respectively, we write $A\#B$ for the $r \times (t+u)$ matrix obtained by *concatenating* A and B: the first t columns of $A\#B$ give A, while the last u columns give B.

The following elementary fact will prove critical in our analysis. It generalizes the fact that when the two by two minors of a matrix vanish and the entries of the first row in the first v columns are 0, then the rest of the entries of the first row kill the elements in the first v columns.

Lemma (killing minors). *Let $A = (a_{ij})$ be a matrix and $1 < v < w$ integers such that the $(k+1) \times (k+1)$ minors of $A|_w$ vanish. Suppose also that $a_{1j} = 0$ for $1 \le j \le v$. Then for $v < j \le w$, a_{1j} kills $I_k(A|_v)$.*

Proof. Fix j and fix a $k \times k$ minor of $A|_v$. If the minor involves the first row of A, it is 0, since the first row of $A|_v$ is 0. Therefore we may assume that the minor involves k rows of A other than the first and k columns of A that are actually columns of $A|_v$. Let B denote the $k \times k$ submatrix of A determined by these rows and columns. Consider the $(k+1) \times (k+1)$ submatrix of A that involves, additionally, the first row of A and the j th column of A. This submatrix has the block form $\begin{pmatrix} 0 & a_{1,j} \\ B & C \end{pmatrix}$ where 0 denotes a $1 \times k$

block and C denotes a $k \times 1$ block. The determinant of this matrix is 0 by hypothesis, and is equal to $\pm a_{i,j} \det(B)$. The result follows. $\square$

Now suppose that we want to create a family of ideals that can be used to prove that ideals of the form $I_3(X)$ are prime. If we kill the variables in the first v columns of the first row, we are led to consider ideals in which the 2×2 minors of the first v columns are 0 and the 3×3 minors of the entire matrix are 0. In addition, some of the entries of the first row are 0. Eventually we may lose the entire first row.

When we consider building an appropriate family for $I_4(X)$, we are led to consider ideals of the form $I_3(X|_v) + I_4(X) + I_a(X)$. But once the first row is gone, and we start to kill entries of te second row, we see that we need to consider ideals of the form $I_2(X|_u)+I_3(X_v)+I_4(X)+I_a(X)$. This suggests studying the large class that we are about to introduce.

Let X be an $r \times s$ matrix and $1 \leq t \leq \min r,\, s$ as before. Let $\sigma = (s_0,\, s_1,\, \ldots\, , s_{t-1})$, where the s_j are nonnegative integers $\leq s$ and $s_{t-1} = s$. We denote by $I_\sigma(X)$ the ideal

$$I_1(X|_{s_0}) + I_2(X|_{s_1}) + \cdots + I_t(X|_{s_{t-1}}).$$

We shall prove:

Theorem. *Let K be a field, and X an $r \times s$ matrix of indeterminates over K with r, s, t as above. Then all of the ideals $I_\sigma(X) + I_{k,h,a}(X)$ are radical, where $\sigma = (s_1,\, \ldots\, , s_{t-1})$ as above.*

We shall also prove that certain ideals among these are prime, and the the quotients by these primes are Cohen-Macaulay, but before we state the precise result, we want to introduce some restrictions on the elements σ and k, h, a used to describe the ideals.

Exactly as in our analysis of the case where $t = 2$, if $k > 0$ or $h > 0$ we can consider instead an ideal of the same type defined using a matrix of indeterminates with at least one dimension strictly smaller than r or s. Henceforth, we assume that $h = k = 0$. Having a positive value for s_0 has the same effect as having the same value for k. We may likewise assume that $s_0 = 0$. If we are not killing any $j \times j$ minors in our sum $I_\sigma(X)$, we assume that $s_{j-1} = j - 1$. Note that $X|_{j-1}$ has rank at most $j - 1$ automatically. Also note that if the size j minors vanish for the first u columns, the same is true for the size $j + 1$ minors. This enables us to assume that $s_{j-1} \leq s_j$. But we can say more: the size $j + 1$ minors will vanish for $X|_{u+1}$ as well, since a $j + 1$ size minor that involves the last column may be expanded with respect to that column, and the cofactors are size k minors of $X|_u$. Henceforth, we may assume without loss of generality that $0 < s_1 < s_2 \cdots < s_{t-1} = s$. When this condition holds, we shall say that σ is *standard*. Note that when we want to work with $I_t(X)$, we work instead with $I_\sigma(X)$ for $\sigma = (0, 1, 2, 3, \ldots\, , t - 2, s)$.

We can now state a more precise version of the theorem that we are aiming to prove.

Theorem. *Let K, X, r, s, t, k, h, σ, and a be as above. Then $I_\sigma(X) + J_{k,h,a}(X)$ is radical.*

If σ is standard and $a = s_k$ for some k (the case where $k = 0$, when $s_k = 0$, is included), then $P = I_\sigma(X) + I_{s_k}(X)$ is prime, and the ring $K[X]/P$ is Cohen-Macaulay.

The proof will occupy as for a while, but is, in fact, quite similar to the argument for the case where $t = 2$.

We first prove:

Lemma. *Let $0 \le k < t \le r$ be integers and K a field. Let L be a nonzero linear functional on K^r and let $0 = V_0 \subseteq V_1 \subseteq \cdots \subseteq V_{t-1}$ be a nondecreasing chain of subspaces of K^r such that L vanishes on V_k (hence, on all of $V_1, \ldots, V_k$) and $\dim_K(V_j) \le j$ for $1 \le j \le t-1$. Then there exists a chain of subspaces $0 = W_0 \subseteq W_1 \subseteq \cdots \subseteq W_{t-1}$ in K^r such that L vanishes on W_k, $V_j \subseteq W_j$ for $1 \le j \le t-1$, and $\dim(W_j) = j$ for $1 \le j \le t-1$.*

Proof. We construct the W_j by reverse induction on j. We may evidently choose $W_{t-1} \subseteq K^r$ such that $V_{t-1} \subseteq W_{t-1}$ and $\dim_K(W_{t-1}) = t-1$, since $\dim(V_{t-1}) \le t-1 < r$. If $W_{j+1}, \ldots, W_{t-1}$ have already been chosen satisfying the required conditions, $j > 1$, then there are two cases. If $j \ne k$, simply chose W_j of dimension j lying between $V_j \subseteq W_{j+1}$, which is possible since $\dim(V_j) \le j$ and $\dim(W_{j+1}) = j+1$. If $j = k$, let H denote the kernel of L, a codimension one subspace of K^r. We now have to choose W_k of dimension k so that it contains V_k and is contained in $H \cap W_{k+1}$. But the dimension of $H \cap W_{k+1} \ge \dim(H) + \dim(W_{k+1}) - r = r - 1 + k + 1 - r = k$, and since $V_k \subseteq H \cap W_{k+1}$ and has dimension at most k, this is possible. $\square$

We can now show the irreducibility of the algebraic sets corresponding to the ideals we are claiming to be prime.

Proposition. *With notation as in the Theorem, if σ is standard, $V = V(I_\sigma(X) + J_{s_k}(X))$ is irreducible.*

Proof. Consider $r \times (t-1)$ matrices B such that the first k entries of the first row are 0. These may be thought of as the points of $\mathbb{A}_K^{r(t-1)-k}$. Let C_j be a $j \times (s_j - s_{j-1})$ matrix over K, $1 \le j \le t-1$. (Recall that $s_0 = 0$ and $s_{t-1} = s$.) Consider the matrix

$$A = B|_1 C_1 \# B|_2 C_2 \# \cdots \# B|_{t-1} C_{t-1}.$$

The first k columns are in the span of the columns of $B|_k$ and so all have a 0 as their initial entry. Moreover, the columns of $A|_{s_j}$ are in the span of the columns of $B|_j$ for every j, and so the rank of $A|_{s_j}$ is at most j for every j. That is, A is a point of V. The choices for C_j are parametrized in bijective fashion by the points of $\mathbb{A}_K^{j(s_j - s_{j-1})}$ for all j. Therefore, we have a map $\mathbb{A}_K^N \to V$, where

$$N = r(t-1) - k + \sum_{j=1}^{t-1} j(s_j - s_{j-1}).$$

To show that V is irreducible, it suffices to show that this map is onto.

Consider any matrix A representing a point of V. Let V_j be the span of the columns of $A|_{s_j}$. Then the V_j satisfy the conditions of the Lemma, and we may choose W_j as in the lemma: the linear functional is projection on the first coordinate. Choose B so that its first column spans W_1, its first two columns span W_2, and, in general, its first j columns span W_j. It is a straightforward induction to prove that this can be done.

Now the columns of $A|_{s_j}$ are in the span of the columns of $B|_j$ for all j: in particular, this is true for the last $s_j - s_{j-1}$ columns, which says precisely that the matrix formed from those columns has the form $B|_j C_j$, as required. $\square$