Math 711: Lecture of September 14, 2005

We can now compute the dimension of V, keeping the above notation. We can consider the open set $U \subseteq V$ where the matrix formed by the columns indexed by the $s_{j-1}+1$, $1 \le j \le t-1$, has rank $t-1$: call this matrix B. Note that U is non-empty because we can use part of the standard basis $e_2, \ldots, e_t$ for K^r for the columns of A indexed by the numbers $s_{j-1}+1$, $1 \le j \le t-1$, and take the rest of the columns of A to be 0.

For each j, the submatrix D_j of A consisting of the columns indexed by $s_{j-1}+1, \ldots, s_j$ can be written uniquely as a linear combination of the columns of $B|_j$. The coefficients needed comprise the columns of a $j \times (s_j - s_{j-1})$ matrix C_j. Note that the first column of C_j is the last column vector in the standard basis for K^j: this corresponds to the fact that the first column of D_j is the same as the column of A indexed by $s_{j-1}+1$ and is the j th column of B. It is therefore the last column of $B|_j$. The entries of C_j other than the first column are arbitrary scalars and therefore C_j may be thought of as varying in an affine space $A^{j(s_j - s_{j-1}-1)}$, and this is also true, therefore, of D_j. It follows that the dimension of V should be

$$r(t-1) - k + 1(s_1 - 1) + 2(s_2 - s_1 - 1) + \cdots + (t-1)(s_{t-1} - s_{t-2} - 1)$$

which we can rewrite as

$$r(t-1) - k - (s_1 + s_2 + \cdots + s_{t-2}) + (t-1)s - \binom{t}{2}$$

.

We can make this more precise as follows. Let $W \subseteq \mathbb{A}_K^{r(t-1)}$ be the non-empty open set consisting of matrices of rank $t-1$, and let $f : U \to W$ be the map that sends the matrix A to the matrix $B = f(A)$ consisting of the columns of A with indices $s_{j-1}+1$, $1 \le j \le t-1$. For fixed B, consider the fiber of f over B. Let V_j be the vector space spanned by the first j columns of B. Then the fiber may be described as consisting of all matrices A such that each column of A indexed by $s_{j-1}+1$, $1 \le j \le t-1$, is the j th column of B and each column of A with index h, $s_{j-1}+1 < h \le s_j$ is in the vector space V_j. It follows that the fiber is isomorphic with

$$\prod_{j=1}^{t-1} V_j^{s_j - s_{j-1} - 1},$$

so that each fiber has dimension $d = \sum_{j=1}^{t-1} j(s_j - s_{j-1} - 1)$, and so the dimension of U (and, likewise, of V) is the sum of the dimensions of W and d, as required. We have now proved:

Theorem. *With notation as above, if σ is standard then*

$$\dim\big(V(I_\sigma(X) + J_{s_k}(X))\big) = (r+s)(t-1) - k - (\sum_{j=1}^{t-2} s_j) - \binom{t}{2}. \quad \square$$

In our discussion of dimension we used a fact relating the dimension of the domain of a dominant morphism to the dimension of the image and the dimension of a "typical" fiber. We treat this result formally below, but we first need some important facts about flatness. Part (a) is the first Lemma in the Lecture of December 8 from the notes for Math 711, Fall 2004. Part (b) is part (d) of the first Lemma in the Lecture of October 11, Math 711, Fall 2004.

Lemma. *Let A, R, and S be Noetherian rings.*

(a) *(Generic freeness) Let A be a domain. If M is a finitely generated module over a finitely generated A-algebra R, then there exists $a \in A - \{0\}$ such that M_a is free over A_a.* $\square$

(b) *If $(R, m, K) \to (S, n, L)$ is a flat local homomorphism of local rings, then* $\dim(S) = \dim(R) + \dim(S/mS)$. $\square$

We are now ready to prove the result we need:

Proposition. *Let $f : V \to W$ be a dominant morphism of algebraic varieties over an algebraically closed field K. Suppose that there is a dense open subset W_0 of W such that for every point $w \in W_0$, the fiber $f^{-1}(w)$ has dimension d. Then* $\dim(V) = \dim(W) + d$.

Proof. We may replace W by an affine open set contained in W_0 and V by an open affine set contained in $f^{-1}(W_0)$. Thus, we may assume that V and W are affine and the dominance of f is then equivalent to the injectivity of the map of domains $K[W] \hookrightarrow K[V]$. By the Lemma of Generic Freeness, after localizing at one element of $K[W] - \{0\}$ we may assume that the map is flat. Consider any maximal ideal m of $K[W]$. The points of the fiber over m correspond to the maximal ideals of $K[V]/mK[V]$ and we can choose a maximal ideal n of $K[V]$ lying over m such that $\dim(K[V]/mK[V])_n = d$. But $\dim(V) = \dim(K[V]) = \dim(K[V]_n)$ and $\dim(W) = \dim(W)_m$. The result now follows from part (b) of the Theorem applied to the flat local map $K[W]_m \to K[V]_n$. $\square$

It is, moreover, always true that given a dominant map of varieties $V \to W$ there does exist an open set W_0 on which the dimension of the fiber is constantly $\dim(V) - \dim(W)$. Cf. the Lemma on p. 3 of the Lecture Notes of September 8, Math 711, Fall 2004.

We can now complete the proof that all the ideals of the form $I = I_\sigma(X) + J_a(X)$ are radical. Because of our result on irreducibility, this also shows that the ones where $a = s_k$ are prime. As usual we may assume $a < s$ or else we can work with the matrix obtained by deleting the first row of X instead. Let $x = x_{1,a+1}$. We use the two lemmas that are the basis for the method of principal radical systems. If we specialize x to 1 and all other entries of the matrix to 0 we see that we have a point A where all generators of I vanish but x does not. Thus, $I + (x) = I_\sigma + J_{a+1}(X)$ is strictly larger than I, and therefore radical by the induction hypothesis. If $a = s_k$ for some k we are done, since we know that $\mathrm{Rad}(I)$ is then prime. Otherwise we have that $s_k < a < s_{k+1}$ for some k. In this case, from the lemma on killing minors we have that $xI_k(X|_a) \subseteq I$. Let σ' be the t-tuple that agrees with σ except that we change the $k+1$st entry s_k to a. By the induction hypothesis, $I_{\sigma'}(X) + J_a(X)$ is radical and, therefore, prime: call it P, and $I \subseteq P$. But $xP \subseteq I$, and so I is radical. $\square$