Math 711: Lecture of September 19, 2005

Our next objective is to prove the Cohen-Macaulayness assertions in the statement of the second Theorem of the Lecture of September 12. The argument is entirely similar to what we did earlier in studying the ideal generated by the 2×2 minors of a matrix of indeterminates.

We use reverse induction, assuming the result that larger ideals of the form $I_\sigma + J_{s_k}(X)$ are Cohen-Macaulay.

Suppose that a specific prime of the form $I_\sigma + J_{s_k}(X)$ is given. Call the ideal P. To show that $K[X]/P$ is Cohen-Macaulay, it suffices to show that the depth of $K[X]/P$ on the ideal m generated by all the $x_{i,j}$ in $K[X]$ is $d = \dim\,(K[X]/P)$: we review the relevant facts about the Cohen-Macaulay property below. Let $x = x_{1,s_k+1}$. Since we already know that $K[X]/P$ is a domain, we have that x is a nonzerodivisor, and so $K[X]/P$ is Cohen-Macaulay if and only if $K[X]/(P + xK[x])$ is, and this may be described as $K[X]/(I_\sigma(X) + J_{s_k+1}(X))$.

There are two cases. If $s_k+1 = s_{k+1}$, then $I+xK[X]$ is a larger prime ideal of our family, and so killing it gives a Cohen-Macaulay ring by the induction hypothesis. If $s_k+1 < s_{k+1}$ then $I + (x)$ is radical. By the lemma on killing minors, each of the variables $x_{1,b}$ for $s_k+1 < b < s_{k+1}$ kills $I_k(X|_{s_k+1})$. Let σ' be the result of changing s_k in σ to s_k+1, while leaving all other entries fixed. Let $Q_1 = I_{\sigma'} + J_{s_k+1}(X)$ and $Q_2 = I_\sigma(X) + J_{s_{k+1}}(X)$. Both of these ideals are prime, and we know that they have Cohen-Macaulay quotients by the induction hypothesis. This is also true for $Q_3 = Q_1 + Q_2 = I_{\sigma'}(X) + J_{s_{k+1}}(X)$.

Note that $V(P + (x)) = V(Q_1) \cup V(Q_2)$ by the lemma on killing minors: since all of the ideals are radical, we have that $P = Q_1 \cap Q_2$.

Moreover, $K[X]/Q_1$ has dimension $d-1$: among the numbers used in calculating the dimension, s_k has increased by one while all others, including k, have not changed. Similarly, $K[X]/Q_2$ has dimension $d-1$: here, only k has changed, increasing by 1. Finally, $K[X]/Q_3$ has dimension $d-2$, since in this case k has increased by 1 and s_k has increased by one. Since these are Cohen-Macaulay, in the short exact sequence

$$0 \to K[X]/(P + (x)) \to K[X/Q_1 \oplus K[X]/Q_2 \to K[X]/Q_3 \to 0$$

the depths of the middle and right hand terms on m are $d-1$ and $d-2$ respectively, and so the depth of $K[X]/(P + (x))$ is $d-1$, as required. $\square$

We next review the basic facts about Cohen-Macaulay rings.

$x_1, \ldots, x_n \in R$ is called a *possibly improper regular sequence* on the R-module M if x_1 is not a zerodivisor on M and for every i, $1 < i \leq n$, x_i is not a zerodivisor on $M/(x_1, \ldots, x_{i-1})M$. It is called a *regular sequence* if, moreover, $(x_1, \ldots, x_n)M \neq M$.

Let $R \to S$ be a homomorphim of Noetherian rings, I an ideal of R, and M a finitely generated S-module. Then M is also an R-module by restriction of scalars. If $IM \neq M$ we define the *depth* of M on I or *depth*$_I M$ to be the supremum of lengths of regular sequences in I on M. By a theorem, this supremum is finite, every regular sequence can be extended to a maximal one, and all maximal regular sequences have the same length, which is the depth. If $IM = M$ the depth is defined to be $+\infty$. It turns out the $\mathrm{depth}_I M = \mathrm{depth}_{IS} M$. The most important case is where $S = R$, but it is often useful to have the greater generality available. REFS

By a theorem, if $d = \mathrm{depth}_I M$ and N is any R-module with annihilator I, then $\mathrm{Ext}^i_R(N,\, M) = 0$ for $i < d$ while $Ext^d_R(N,\, M) \neq 0$. In particular, one may choose $N = R/I$, and the long exact sequence for Ext may then be used to prove facts about the behavior of depth in short exact sequences. For example, if $0 \to M' \to M \to M'' \to 0$ is exact, and $\mathrm{depth}_I M'' < \mathrm{depth}_I M$, then $\mathrm{depth}_I M' = \mathrm{depth}_I M'' + 1$. We used this last assertion earlier, when the depth of the middle term was $d - 1$ and the depth of the right hand term was $d - 2$.

A local ring $(R,\, m,\, K)$ is called *Cohen-Macaulay* if some (equivalently, every) system of parameters is a regular sequence in the ring. Regular local rings are Cohen-Macaulay and so are their quotients by an ideal generated by part of a system of parameters: the latter are called *local complete intersections*. If R is a Cohen-Macaulay local ring, then each of its localizations at a prime is again Cohen-Macaulay. REFS

A Noetherian ring R is called *Cohen-Macaulay* if all of its localizations at prime (equivalently, at maximal) ideals are Cohen-Macaulay local rings. The quotient of Cohen-Macaulay ring by a proper ideal generated by a regular sequence is again Cohen-Macaulay. All Noetherian rings of dimension 0 are Cohen-Macaulay, and all reduced Noetherian rings of dimension 1 are Cohen-Macaulay as well. Of course, this includes the one-dimensional domains. By a theorem, all normal Noetherian domains of dimension two are Cohen-Macaulay. REFS

If a local ring R is a module-finite extension of a regular local ring A, then R is Cohen-Macaulay if and only if it is free as an A-module. REF

We want to focus on the graded case. Let R be an $\mathbb{N}$-graded finitely generated algebra over a field K with $R_0 = K$. Let m denote the unique homogeneous maximal ideal: it is spanned by all forms of positive degree, and may also be described as $\bigoplus_{n=1}^{\infty} R_n$. Let $\dim(R) = d$. By a theorem, there is always a sequence $f_1,\, \ldots\,, f_d$ of forms in m such that $\mathrm{Rad}\big((f_1,\, \ldots\,, f_d)\big) = m$. Such a sequence is called a *homogeneous system of parameters*. Note that if K is infinite and R is generated by R_1 over K, then these may be chosen in R_1. In general it is necessary to use forms of degree larger than 1. If $f_1,\, \ldots\,, f_d$ is a homogeneous system of parameters for R, R is module-finite over its subring $K[f_1,\, \ldots\,, f_d]$, and this subring is a polynomial ring, i.e., $f_1,\, \ldots\,, f_d$ are algebraically independent over K.

The image of a homogeneous system of paramaters for R in R_m is a system of parameters for R_m.

Theorem. *R be an $\mathbb{N}$-graded finitely generated K-algebra as above, where K is a field, with $R_0 = K$ and let m be the homogeneous maximal ideal of R. Suppose that* $\dim(R) = d$. *The following conditions are equivalent:*

(1) *R is Cohen-Macaulay.*
(2) *R_m is Cohen-Macaulay.*
(3) *Some (equivalently, every) homogeneous system of parameters for R is a regular sequence in R.*

(4) $\mathrm{depth}_m R = d$.
(5) *For some (equivalently, every) homogeneous system of parameters* $f_1, \ldots, f_d$ *for* R, R *is a free module over its polynomial subring* $K[f_1, \ldots, f_d]$.

We next want to discuss a different approach to proving the Cohen-Macaulay property for rings defined by killing minors of matrices of indeterminates. The approach is to consider systematically a class of graded rings with the property that one can take associated graded rings repeatedly until one has a ring that is simply the quotient of a polynomial ring by an ideal generated by square-free monomials: a *Stanley-Reisner* ring or *face* ring. We shall study these rings first.

By a (finite) simplicial complex Δ with vertices $x_1, \ldots, x_n$ we mean a finite set of subsets of $\{x_1, \ldots, x_n\}$ with the following properties:

(1) $\{x_j\} \in \Delta$, $1 \leq j \leq n$.
(2) If $\sigma \in \Delta$ and $\tau \subseteq \sigma$ then $\tau \in \Delta$.

The elements σ of Δ are called the *simplices* or *faces* of of Δ: the maximal ones determine Δ and are called the *facets* of Δ. The *dimension* of a simplex is one less than its cardinality, and the *dimension* of Δ is the largest dimension of any of its simplices.

Let K be a commutative ring. The main case is where K is a field, but we allow arbitrary rings in the definition. Let I_Δ be the ideal of the polynomial ring $K[x_1, \ldots, x_n]$ generated by all monomials $x_1^{i_1} \cdots x_n^{i_n}$ such that the set $\{x_j : i_j > 0\} \notin \Delta$. Let $K[\Delta] = K[x_1, \ldots, x_n]/I_\Delta$. We shall see that $K[\Delta]$ is a reduced ring of dimension $\dim(\Delta) - 1$ whose minimal primes correspond bijectively to the facets of Δ.