Math 711: Lecture of September 21, 2005

We continue to review some basic facts about simplicial complexes.

The *convex hull* of a finite set $v_1, \dots, v_k \subseteq \mathbb{R}^n$ is the smallest convex set that contains it; it may also be characterized as

$$\{r_1 v_1 + \cdots + r_k v_k : 0 \le r_i \le 1,\ \ r_1 + \cdots + r_k = 1\}.$$

Let $e_1, \dots, e_n$ be the standard basis for $\mathbb{R}^n$. The *geometric realization* $|\Delta|$ of a finite simplicial complex Δ with vertices $x_1, \dots, x_n$ may be defined as the topological subspace of $\mathbb{R}^n$ obtained as the union of the convex hulls of the sets $\{e_{i_1}, \dots, e_{i_k}\}$ such that $\{x_{i_1}, \dots, x_{i_k}\}$ is a face of Δ.

Then the dimension of Δ is the same as the dimension of the real topological space $|\Delta|$.

We also recall the notion of the simplicial homology $H_\bullet(\Delta;\, K)$ of Δ with coefficients in a commutative ring K. We form a free complex $\mathcal{C}_\bullet$ such that $\mathcal{C}_j$, $j \ge 0$ is the free K-module on the free basis consisting of simplices of Δ of dimension j. To define the differential we need only specify its value on a typical j-simplex $\sigma = \{x_{i_1}, \dots, x_{i_{j+1}}\}$. We do this by letting

$$d\sigma = \sum_{t=1}^{j+1} (-1)^{t-1}(\sigma - \{x_{i_t}\}).$$

Then $H_\bullet(\Delta; K)$ is defined as the homology of this complex, and is a topological invariant of $|\Delta|$ and, in fact, only depends on the homotopy type of $|\Delta|$, but the proof of this is outside the scope of these lectures.

We also define the *reduced simplicial homology* $\widetilde{H}_\bullet(\Delta;\, K)$ as the homology of the complex obtained by modifying the complex $\mathcal{C}_\bullet$ above to include a term in degree -1: the additional term is the free module on one generator corresponding to the unique simplex of dimension -1, to wit, the empty simplex $\emptyset$. The value of d on every $\{x_i\}$ is $\emptyset$. Note that $\widetilde{H}_i(\Delta;\, K) \cong H_i(\Delta;\, K)$ for $i \ge 1$. The rank of $\widetilde{H}_0(\Delta;\, K)$ is one less than the rank of $H_0(\Delta;\, K)$. The rank of the latter is the number of connected components of $|\Delta|$. In particular, when $|\Delta|$ is connected, $\widetilde{H}_0(X) = 0$.

Proposition. *Let K be an integral domain and Δ a finite simplicial complex with vertices $x_1, \dots, x_n$. There is a bijection between the facets σ of Δ and the minimal primes of $K[\Delta]$ (equivalently, of I_Δ) in which the minimal prime corresponding to σ is generated by the images of the variables not in σ. The quotient by the minimal prime corresponding to σ may be identified with the polynomial ring over K in variables corresponding bijectively with the vertices of σ.*

Hence, if K is a field, $\dim(K[\Delta]) = \dim(\Delta) + 1$. *More generally,* $\dim(K[\Delta]) = \dim(K) + \dim(\Delta) + 1$.

Proof. The final statement about dimension follows from the analysis of the minimal primes, and the fact that the dimension of $K[x_1, \dots, x_k]$ is $\dim(K) + k$ when K is Noetherian.

We may analyze the minimal primes of $I = I_\Delta$ as follows. Consider any prime P containing I_Δ. Note that for each subset of $x_1, \ldots, x_n$ that is not a face, the product of the variables in that subset is 0, and so at least one of them is in P. But a subset S of the vertices meets every subset that is not a face if and only if its complement is a face. (If the complement were not a face, it would have to meet it, a contradiction. On the other hand, if the complement of the face σ fails to meet a set, the set must be contained in σ, and therefore is a face.) Thus, any prime $P \supseteq I$ contains the variables in the complement of a face, and therefore the variables in the complement of a facet. But the variables in the complement of a facet generated a prime Q that contains I (each generator of I is divisible by one of variables in Q), and there can be no smaller prime that contains I within Q, since it would have to contain the complement of a larger face. $\square$

Note that every quotient ring R of $K[x_1, \ldots, x_n]$ obtained by killing square-free monomials has the form $K[\Delta]$: let Δ consist of all subsets of the variables the image of whose product is not 0 in R.

Think of $x_1, \ldots, x_n$ as coordinate functions on $\mathbb{A}^n_K$, where K is a field. By a coordinate k-plane we mean a k-dimensional vector subspace of $K^n = \mathbb{A}^n_K$ defined by the vanishing of a subset of the coordinate functions: $n-k$ coordinate functions will be needed. Then, with K a field, $V(I_\Delta) \subseteq \mathbb{A}^n_K$ is the union of the coordinate k-planes (where k may vary) defined by the vanishing of the variables in the complements of the facets of Δ. For example, when $n = 3$ and the facets of Δ are $\{x_1, x_2\}$, $\{x_1, x_3\}$, and $\{x_2, x_3\}$, $V(I_\Delta)$ is the union of the three planes $V(x_1)$, $V(x_2)$, and $V(x_3)$.

If the field is the real numbers, and C denotes the standard simplex which is the convex hull of the standard basis $e_1, \ldots, e_n$, then $V(I_\Delta) \cap C$ is the same as the geometric realization $|\Delta|$ of Δ.

We next want to describe Reisner's criterion for when the ring $K[\Delta]$ is a Cohen-Macaulay ring. In order to do so, we introduce the concept of a link within a simplicial complex.

Let Δ be a simplicial complex and $\sigma \in \Delta$. We define the *link* of σ in Δ to be the simplicial complex $\{\tau \in \Delta : \tau \cap \sigma = \emptyset \text{ and } \tau \cup \sigma \in \Delta\}$. One defines the link of a vertex x_i to be the same as the link of $\{x_i\}$. The link of $\emptyset$ is simply Δ. The link of $\sigma = \{x_{i_1}, \ldots, x_{i_k}\}$ may also be obtained recursively as follows: take the link of x_{i_1} in Δ to produce Δ_1. At the the $(t+1)$ st stage take the link of $x_{i_{t+1}}$ in Δ_t to produce Δ_{t+1}. Then Δ_k is the link of σ.

For example, if Δ is the triangulation of the disk obtained from a convex polygon (including its interior) by joining an interior point x to each vertex on the boundary, the link of x in Δ consists of the boundary: it is one-dimensional.

We shall eventually prove:

Theorem (Reisner's criterion). *Let K be a field and Δ a finite simplicial complex. Then $K[\Delta]$ is Cohen-Macaulay if and only if for every Λ which is either Δ itself or a link of a simplex of Δ, if Λ has dimension h then $\widetilde{H}_i(\Lambda; K) = 0$, $0 \le i \le h-1$.*

Mentioning that Δ itself satisfies the condition is redundant here, since it is the link of $\emptyset$, but the statement about Δ is included for emphasis. If $|\Delta|$ is a manifold with boundary, the condition on smaller links is automatically satisfied. Although it is not immediately obvious, the condition given is topological, i.e., it depends only on $|\Delta|$. We note that if Δ

arises from a triangulation of the real projective plane, then $K[\Delta]$ is Cohen-Macaulay if and only if the characteristic of K is different from 2.

We next want to prove that when K is a field, the rings $K[\Delta]$ are reduced, and that if K has positive characteristic p, then $K[\Delta]$ is F-*pure*: we need to explain the meaning of this notion.