Math 711: Lecture of September 23, 2005

We recall that a map of R-modules $f : N \to M$ is called *pure* over R if for every R-module Q, $Q \otimes_R N \to Q \otimes M$ is injective. In particular, $N \to M$ must itself be injective. If $N \to M$ splits, so that N is a direct summand of M, then the inclusion $N \to M$ is pure, i.e., maps of the form $N \to N \oplus Q$ are pure. Purity should be thought of as a weakening of the condition of being a direct summand.

Theorem. *If $N \to M$ is pure and M/N is finitely presented then $N \to M$ splits.*

Since tensor product commutes with direct limit, if one has a direct limit of pure maps (where a map from $N_i \to M_i$ to $N_j \to M_j$ is a commutative diagram

$$\begin{array}{ccc} N_j & \longrightarrow & M_j \\ \uparrow & & \uparrow \\ N_i & \longrightarrow & M_i \end{array} \quad)$$

the direct limit map is also pure. It is obvious that if $N \subseteq M$ is pure then $N \subseteq M_0$ is pure for every M_0 between N and M. By the fact stated for direct limits, we have that $N \subseteq M$ is pure if and only if $N \subseteq M_0$ is pure for every M_0 with $N \subseteq M_0 \subseteq M$ such that M_0/N is finitely generated. When the base ring R is Noetherian, so that M_0/N is finitely presented whenever it is finitely generated, we have that $N \subseteq M$ is pure if and only if $N \subseteq M_0$ splits for every M_0 such that $N \subseteq M_0 \subseteq M$ and M_0/N is finitely generated.

Purity is obviously preserved by base change: this is immediate from the associativity of tensor product.

A map of rings $R \to S$ is *pure* if it is pure as a map of R-modules. We also say that S is a *pure* R-algebra in this case. When $R \hookrightarrow S$ is pure, every ideal I of R is contracted from S: this is because when one tenors $R \to S$ with R/I, the resulting map $R/I \to S/IS$ is injective.

The composition of two pure maps is pure. If we have a ring homomorphism $R \to S$ and an S-linear map $M \to N$ that is pure over S, then it is also pure over R when we restrict scalars to R. We leave this as an exercise. If we have algebras $R \to S \to T$ such that $R \to S$ is pure and $S \to T$ is pure, then $R \to T$ is pure, since $S \to T$ is also pure as a map of R-modules, and we may compose.

A ring R of prime characteristic $p > 0$ is called *F-pure* (respectively, *F-split*) if the Frobenius endomorphism $F : R \to R$ is pure (respectively, split). Here, for $r \in R$, $F(r) = r^p$. Since we may compose, R is F-pure (respectively, F-split) if and only if all the iterations F^e are pure (respectively, split). Of course, if R is F-split then it is F-pure. Note that the Frobenius endomorphism is injective if and only if R is reduced.

Let $q = p^e$. Then the expansion of the ideal I under $F^e : R \to R$ (note that $F^e(r) = r^q$) is denoted $I^{[q]}$: it is the ideal of R generated by all q th powers of elements of I, and it is also generated by the q th powers of the elements in any set of generators of I.

Note that if R is F-pure and $r \in R$ is such that $r^q \in I^{[q]}$, where $q = p^e$, then $r \in I$. This is equivalent to the fact that I is contracted from the second copy of R under $F^e : R \to R$.

Note that if K is a field, any nonzero map $K \to S$ is split over K and, hence, every nonzero K-algebra is pure. The point is that the image of $1 \in K$ may be extended to a vector space basis for S over K, and there is a K-linear map $S \to K$ that has the value 1 on 1 and is 0 on the rest of the basis (the specification of values on the rest of the basis does not matter: any specification gives the required splitting).

It will be convenient to refer to the *support* of a monomial as the set of variables occurring with a positive exponent in the monomial.

Proposition. *Let K be a field of characteristic $p > 0$ and let Δ be a finite simplicial complex. Then $K[\Delta]$ is F-split and, hence, F-pure.*

Proof. It is clear that F is injective, since the image of $\sum_\mu c_\mu \mu$ as μ runs through some finite set of monomials with support in Δ and each $c_\mu \neq 0$ is $\sum_\mu c_\mu^p \mu^p$. The key point is that support of a monomial does not change when one takes its p th power. The image of the map is therefore spanned over $K^p = \{c^p : c \in K\}$ by the set of monomials $\mathcal{M}$ with support in Δ such that every exponent is divisible by p. Let $\mathcal{N}$ be the set of monomials with support in Δ but such that some exponent is not divisible by p. We know that $K^p \to K$ splits over K^p. Call the splitting θ. We now define a K^p-linear map $K[\Delta] \to (K[\Delta])^p = \mathrm{Span}_{K^p}\mathcal{M}$ by sending $K\mu$ to 0 if $\mu \in \mathcal{N}$ and sending $c\mu$, where $c \in K$ and $\mu \in \mathcal{M}$, to $\theta(c)\mu$. This map is clearly K^p-linear and a retraction. Linearity over $K[\Delta]^p$ is easy to verify using the additional fact that the product of a monomial in $\mathcal{M}$ and a monomial in $\mathcal{N}$ is in $\mathcal{N}$. $\square$

We next want to define Hodge algebras and the subclass of Hodge algebras called algebras with straightening law. But before we give the rather long and technical definition, we want to give an example of an important class of rings which turn out to be Hodge algebras.

Let K be a field, let $1 \leq r \leq s$ be integers, and let X denote an $r \times s$ matrix of indeterminates over K. We shall denote by $K[X/r]$ the subring of the polynomial ring $K[X]$ generated by the $r \times r$ minors of X. If $\underline{j} = (j_1, \ldots, j_r)$ where $1 \leq j_1 < \cdots < j_r \leq s$ are integers, one can form an $r \times r$ submatrix $Y_{\underline{j}}$ of X, and the determinants of these $\binom{s}{r}$ submatrices generate $K[X/r]$.

This ring is the homogeneous coordinate ring of the Grassmann variety of r-dimensional vector subspaces of K^s. The corresponding projective variety parametrizes such subspaces. (Given an r-dimensional subspace W of K^s one may choose a basis for W and make the basis elements into the rows of an $r \times s$ matrix M whose row space is W. The $r \times r$ minors of this matrix, at least one of which is not 0, form the *Plücker coordinates* of the subspace. If one chooses a different basis for W, the new matrix obtained has the form AM where A is an invertible $r \times r$ matrix, and the Plücker coordinates are all multiplied by $\det(A) \neq 0$, so that the Plčker coordinates give a well-defined point of $\mathbb{P}_K^{\binom{s}{r}-1}$. These points obviously satisfy the same relations that are satisfied by the minors of a matrix of indeterminates. Connversely, by a theorem, any $\binom{s}{r}$ scalars satisfying those relations do arise as Plücker coordinates of some subspace.)

Let H be the set of $r \times r$ minors of X. We partially order H by $Y_{\underline{j}} \le Y_{\underline{k}}$ precisely if $j_t \le k_t$, $1 \le t \le r$. It turns out that the monomials in the minors such that the set of minors occurring is linearly ordered under this ordering of H is a basis for $K[X/r]$ over K. Moreover, the product of any two incomparable minors h and h' is a K-linear combination of products of pairs of comparable minors $\alpha\beta$ such that $\alpha < h$ and $\alpha < h'$. We shall prove all of this eventually: for the moment just want to have one concrete illustration for the axioms we will soon introduce.

Note that monomials in the elements of H can be indexed by $\mathbb{N}$-valued functions on H: the function corresponding to a given monomial u assigns to each $h \in H$ the exponent with which it occurs in u.

A non-empty subset Σ of $\mathbb{N}^H$ is called a *semigroup ideal* if whenever $h \in \Sigma$ and $h' \in \mathbb{N}^H$ then $h + h' \in \Sigma$. By a *generator* of a semigroup ideal Σ we simply mean a minimal element of Σ, i.e., one which cannot be written $h + h'$ for $h \in H$ and $h' \in \mathbb{N}^H - \{0\}$. Any Hodge algebra will have a set of standard monomials that are a basis: the exponents that correspond to other monomials will form a semigroup ideal. In the case of $K[X/r]$, Σ consists of all of elements of $\mathbb{N}^H$ that are nonzero on two distinct incomparable elements of H.

We are now ready to give our axiomatized generalization of this set-up. A word of caution: in the literature, Hodge algebras and algebras with straightening law (ASLs) are defined without necessarily being graded. However, this level of generality has not proved very useful. We shall work entirely in the graded case, and so for us the terms "Hodge algebra" and "algebra with straightening law" are synonymous with "graded Hodge algebra" and "graded algebra with straightening law" as used by some authors.

Let K be a ring, R an $\mathbb{N}$-graded K-algebra, and let $H \subseteq R$ be a finite set of forms of positive degree such that $R = K[H]$. Assume that a partial ordering $\le$ on H is given, and a semigroup ideal $\Sigma \subseteq \mathbb{N}^H$. If $c \in \mathbb{N}^H$, we write c_h for $c(h)$ and $\underline{h}^c$ for the monomial $\prod_{h\in H} h^{c_h}$. We say that $\underline{h}^c$ is the monomial *corresponding* to c and that c is the *exponent* of $\underline{h}^c$ (this notion turns out to be well-defined in important cases because of axiom (1) below). We refer to the set $\{h \in H : c_h \neq 0\}$ as the *support* of c and also as the *support* of $\underline{h}^c$, in cases where it is unique (again, see the comments in (1) below). We say that R is a *Hodge algebra* over K on H *governed by* Σ if the following two axioms hold:

(1) The function $\mathbb{N}^H \to R$ sending c to $\prod_{h\in H} h^{c_h}$ is injective on $\mathbb{N}^H - \Sigma$, and its values are linearly independent over K. (The values are called the *standard* monomials. Note that a standard monomial u has a unique support coming from its representation $\underline{h}^c$ using $c \in \mathbb{N}^H - \Sigma$, and we always mean this support when talking about the support of a standard monomial)

(2) If $v = h^c$ is a monomial corresponding to an exponent c that is a generator of Σ, it has a representation

$$(*) \quad v = \sum_{u\in\mathcal{M}} \lambda_u u$$

where $\mathcal{M}$ is a finite set of standard monomials (it may be empty, in which case the right hand side is interpreted as 0), every $\lambda_u \in K - \{0\}$, and such that for every u and every h

in the support of c, for some element h' in the support of u, $h' < h$.

The representations given by (2) are called the *straightening relations* of the Hodge algebra. A Hodge algebra is called *discrete* if all the right hand sides are 0 in (2). A discrete Hodge algebra is isomorphic with the a quotient of a polynomial ring $K[h : h \in H]$ by the ideal generated by all the monomials $\{\underline{h}^c : c \in \Sigma\}$. In particular, the face rings $K[\Delta]$ are discrete Hodge algebras.

A key point in the theory of Hodge algebras is that one may take associated graded rings repeatedly, until one eventually obtains a discrete Hodge algebra. Properties of the discrete Hodge algebra so obtained can, in many cases, be shown to hold for the Hodge algebra itself. In particular, this holds for the Cohen-Macaulay property. We shall use this method to prove that the ring $K[X/r]$ is Cohen-Macaulay, and to give a new proof that rings of the form $K[X]/I_t(X)$ are Cohen-Macaulay.

A Hodge algebra is called an *algebra with straightening law* or *ASL* if Σ consists of all elements of $\mathbb{N}^H$ whose support contains at least two incomparable elements. In this case, $\mathbb{N}^H - \Sigma$ consists of all functions whose support is a linearly ordered subset of H: the corresponding standard monomials are the monomials involving a linearly ordered subset of H. We shall prove that $K[X/r]$ is an ASL. In an ASL, the straightening law permits a stronger conclusion: since each monomial u occurring on the right hand side is a product of elements in a linearly ordered subset of H, there is a least element of H that occurs in each such u, and this element must be less than every element in the support of c, where $v = \underline{h}^c$ in $(*)$ of (2).