Math 711: Lecture of September 26, 2005

Note that the straightening relations for a Hodge algebra are homogeneous: if the terms on the right do not all have the same degree as v, fix a degree different from that of v for which there are nonzero terms of that degree. Then the sum of all the terms of that degree on the right must be zero, a contradiction, for the standard monomials must be linearly indepenendent over K.

If $c \in \mathbb{N}^h$ we write c_h instead of $c(h)$, so that the monomial $\underline{h}^c$ corresponding to c is $\prod_{h \in H} h^{c_h}$.

No matter what the base ring K is, there is a bijection between the semigroup ideals $\Sigma \subseteq \mathbb{N}^H$ and the nonzero monomial ideals of $K[H]$, the polynomial ring with the elements of H as the variables. This bijection takes Σ to the ideal which is the K-span of the monomials $\underline{h}^c$ for $c \in \Sigma$. Conversely, if I is any nonzero ideal generated by monomials in $K[H]$, I is the K-span of the set of monomials contained in I, and the set of exponents of the monomials in I is a semigroup ideal in $\mathbb{N}^H$. A minimal set of monomial generators for such an ideal I is unique, and the exponents correspond to the generators of Σ in the sense defined earlier. Since all of this holds when K is Noetherian, it follows that the set of generators of Σ is finite. Of course, it is also possible to prove this without using any ring theory.

If H is a poset and $h \in H$, we define $\dim(h)$ as the supremum of lengths t of strictly ascending chains $h = h_0 < h_1 < \cdots < h_t$ with h as the first element. (For example, the dimension of a prime ideal P in the poset of prime ideals of a ring R is the same as the Krull dimension of R/P.) Note that in a finite poset H, if $h_1 < h_2$ then $\dim(h_1) > \dim(h_2)$, since a longest chain starting with h_2 will yield a chain of length one more if we insert h_1 at the beginning.

We next note the following:

Theorem. *Let R be a Hodge algebra on H over K governed by Σ. Then the standard monomials are a free K-basis for R, and R is the quotient of a polynomial ring in variables corresponding to the elements of H by the ideal generated by the straightening relations. More precisely, if we introduce an indeterminate X_h over K for every element of H then R is the quotient of $K[X_h : h \in H]$ by the ideal J generated by the straightening relations, each translated into a relation on the indeterminates X_h: for each exponent c that is a generator of Σ, we include one generator*

$$\prod_{h \in H} X_h^{c_h} - \sum_{u \in \mathcal{M}} \lambda_u \prod_{h \in H} X_h^{c_h^{(u)}}$$

in J, where $c^{(u)} \in \mathbb{N}^H - \Sigma$ is the exponent corresponding to u.

Proof. Let d be the supremum of the sums $\displaystyle\sum_{h \in H} c_h$ for c that is a generator of Σ, and let the *weight* of an element $h \in H$ be defined as $(d+1)^{\dim(h)}$. Extend the weight function to the exponent c' by letting the weight of c' be

$$\sum_{h \in H} c'_h \mathrm{weight}(h).$$

Then, for any two exponents c_1, c_2, the weight of their sum is the sum of their weights.

Assume the result is false and that some monomial of degree N cannot be expressed as a linear combination of standard monomials as a consequence of the straightening relations. Choose such a monomial whose exponent is of greatest weight. This monomial v' can be written as vw where v corresponds to an exponent c that is a minimal generator of Σ. It suffices to show that v' is a linear combination of monomials of degree N corresponding to exponents of greater weight.

It will therefore be enough to prove this for v, since multiplying by w will preserve the necessary qualities of the relation: the weight of the exponent for each monomial is increased by the weight of the exponent chosen for w.

Consider the straightening relation

$$v = \sum_{u \in \mathcal{M}} \lambda_u u$$

where each of the standard monomials u that occurs has the same degree as v. It will suffice to show that the weight of the exponent $c^{(u)}$ of each u that occurs is strictly larger than the weight of v. Recall that $v = \underline{h}^c$. Choose h_0 in the support of c of largest possible dimension: call this dimension b. Then the weight of c is

$$\sum_{h \in H} c_h (d+1)^{\dim(h)}$$

and whenever $c_h \neq 0$ we have that $\dim(h) \leq b$, which shows that the sum is bounded by

$$(\#) \quad \sum_{h \in H} c_h (d+1)^b = (\sum_{h \in H} c_h)(d+1)^b \leq d(d+1)^b < (d+1)^{b+1}.$$

Now consider the weight of $u = \underline{h}^{c^{(u)}}$ occurring in the straightening relation. Then $c^{(u)}$ is supported at an element $h_1 \in H$ such that $h_1 < h_0$. Then the weight of $c^{(u)}$ is at least the weight of h_1, which is $(d+1)^{\dim(h_1)}$. Since $\dim(h_1) > \dim(h_0) = b$, this is $\geq (d+1)^{b+1}$, which we showed in (#) above is strictly larger than the weight of c, as required. □

We shall say that two Hodge algebbras have *the same data* if their posets H and H' are isomorphic, say $\phi : H \cong H'$ is the order isomorphism, in such a way that the induced semigroup isomorphism $\mathbb{N}^H \cong N^{H'}$ carries the semigroup ideal Σ for the first Hodge algebra isomorphically onto the semigroup ideal Σ' for the second Hodge alagebra. The simplest case is when $H = H'$ and $\Sigma = \Sigma'$.

We define the*indiscrete part* $\mathrm{Ind}(R)$ of the Hodge algebra R over K on H governed by Σ to be the subset of elements $h \in H$ such that h is in the support of a standard monomial

u occurring with nonzero coefficient in a straightening relation for R over K. Thus, R is discrete if and only if the indiscrete part $\mathrm{Ind}(R)$ of R is empty.

Our strategy is the following: we shall show that if R is not discrete and h_0 is minimal in $\mathrm{Ind}(R)$, then, with $I = h_0R$, the associated graded ring $\mathrm{gr}_I(R)$ is a Hodge algebra over K on H governed by Σ such that

$$\mathrm{Ind}\big(gr_I(R)\big) \subseteq \mathrm{Ind}(R) - \{h_0\}.$$

Once we have shown this it follows immediately that one may form successive associated graded rings in this way until one reaches a discrete Hodge algebra. Technically, the description above is not fully accurate: the poset and semigroup at each stage are the same as the ones at the previous stage only up to an obvious isomorphsim. However, we do reach a discrete Hodge algebra over K with the same data. This will enable us to prove substantial theorems about the properties of R by proving corresponding properties for the discrete Hodge algebra instead.