Math 711: Lecture of September 28, 2005

The following fact will prove critical:

Lemma. *Let R be a Hodge algebra with notation as above, let h_0 be minimal in* $\mathrm{Ind}(R)$, *and let $I = h_0R$. Then for every positive integer j, I^jR is the free K-module spanned by those standard monomials $\underline{h}^c$, $c \in \mathbb{N}^H - \Sigma$, such that $c_{h_0} \geq j$, i.e., such that the standard monomial has h_0^j as a factor.*

Proof. Since R is spanned over K by the standard monomials it is clear that $I^j = h_0^jR$ is spanned over R by the elements $h_0^j\underline{h}^c$ for $c \in \mathbb{N}^H - \Sigma$. We claim that each such monomial is either 0 or else is standard itself. Let e denote the element of $\mathbb{N}^H$ that is 1 on h_0 and 0 elsewhere. If $c + je$ does not yield a standard monomial, we have that $c + je$ is in Σ, and so is the sum of a standard generator c' and some element $f \in \mathbb{N}^H$, i.e., $c + je = c' + f$. We claim that c' must have a positive value on h_0: if not, then the value of f on h_0 is at least j, and we can write $c = c' + (f - je)$, which shows that $c \in \Sigma$, a contradiction.

Now consider a straightening relation for $\underline{h}^{c'}$: since c' is positive on h_0, each nonzero term $\lambda_u u$ on the right involveds a standard monomial u whose support contains an element of H, call it h_0', with $h_0' < h_0$. Since h_0 is minimal in $\mathrm{Ind}(R)$, such an h_0' does not exist, and it follows that $\underline{h}^{c'} = 0$, and so $h_0^j\underline{h}^{c'} = 0$ as well. $\square$

We keep the notations of the Lemma in force, with h_0 a fixed minimal element of $\mathrm{Ind}(R)$. If r is nonzero element of R, there is a largest integer $j \geq 0$ such that $r \in I^j$ (by a degree argument, for example), and then $r \in I^j - I^{j+1}$. We call j the *order* of r with respect to I (or with respect to h_0) and use $\mathrm{ord}_I(r) = \mathrm{ord}\,(r)$ to denote this order. It is clear that $\mathrm{ord}\,(rs) \geq \mathrm{ord}\,(r) + \mathrm{ord}\,(s)$. This generalizes by induction to the case of a product of several elements. If $\mathrm{ord}\,(r) = j$ we write r^* for the element $r + I^{j+1} \in I^j/I^{j+1} \subseteq \mathrm{gr}_I(R)$: r^* is called the *leading form* of r. Then if $\mathrm{ord}\,(rs) = \mathrm{ord}\,(r) + \mathrm{ord}\,(s)$, we have that $(rs)^* = r^*s^*$. Again, this generalizes by induction to the case of a product of several elements. We can now state an important consequence of the Lemma above.

Corollary. *With notation as above, I^j/I^{j+1} has a free K-basis the elements u^*, where u is standard monomial whose exponent has value j on h_0. Thus, the elements u^*, as u runs through all standard monomials, are a free K-basis for* $\mathrm{gr}_I R$.

The order of h_0 is 1, while for all $h \in H - \{h_0\}$, $\mathrm{ord}\,(h) = 0$. *Moreover, for any standard monomial u with exponent c, $u^* = \prod_{h \in H} h^{*c_h}$.* $\square$

Keeping these notations, we next observe that we can give $\mathrm{gr}_I R$ the structure of a graded algebra (not the one come from the fact that it is an associated graded ring) as follows. We simply let the graded piece in degree i be the K-span of all the u^* such that u is a standard monomial of degree i in R. We need to check that if u and u' are standard monomials of degrees i and i', respectively, then $u^*u'^*$ is a K-linear combination of elements v^* where v is standard of degree $i + i'$.

Suppose that u has order m with respect to I and v has order m'. Then then $u^*u'^*$ in $\mathrm{gr}_I R$ is the class of uu' in $I^{m+m'}/I^{m+m'+1}$, by the definition of multiplication for $\mathrm{gr}_I(R)$.

But uu' will be a K-linear combination of standard monomials each of which is a multiple of $h_0^{m+m'}$, and each of which has degree $i+i'$. Say

$$uu' = \sum_{w\in\mathcal{M}} \lambda_w w.$$

It follows that

$$u^*u'^* = \sum_{w\in\mathcal{M},\, \mathrm{ord}_I(w)=m+m'} \lambda_w w^*,$$

and since every w has degree $i+i'$, we have proved what we need.

We now have that $\mathrm{gr}_I(R)$ is an $\mathbb{N}$-graded K-algebra generated by $H^* = \{h^* : h \in H\}$. We use the bijective map $h \mapsto h^*$ to give a partial ordering on H^*, and we let Σ^* be the image of Σ under the obvious isomorphism $\mathbb{N}^H \cong N^{H^*}$ induced by the order isomorphism $H \cong H^*$. We claim that, with the grading described just above, $\mathrm{gr}_I R$ is a Hodge algebra over K on H^* governed by Σ^*, and that

$$\mathrm{Ind}(\mathrm{gr}_I R) \subseteq \{h^* : h \in \mathrm{Ind}(R)\} - \{h_0^*\}.$$

To see this, suppose that $c \in \Sigma$ is a generator, that v is the element with this exponent in R, and that w is the corresponding element in $\mathrm{gr}(I)$ obtained from H^* by using the same exponent. Then we have a straightening relation

$$(*) \quad v = \sum_{u\in\mathcal{M}} \lambda_u u$$

in R. We want to show that from $(*)$ we can derive a corresponding relation in $\mathrm{gr}_I R$, and we need to compare the indiscrete parts. There are two cases. If c is positive on h_0 then the right hand side of $(*)$ must be 0, and we see that we have the relation $w = 0$ in $\mathrm{gr}_I(R)$. No elements from H^* are used on the right. Now suppose that c is 0 on h_0, so that v is a product of elements of H of order 0. Then in $\mathrm{gr}_I R$ we have the relation:

$$(*') \quad w = \sum_{u\in\mathcal{M},\, \mathrm{ord}_I(u)=0} \lambda_u u^*,$$

and all the elements of H^* occurring as factors of the various u^* are in

$$\{h^* : h \in \mathrm{Ind}(R)\} - \{h_0^*\},$$

which is exactly what we need. We have therefore proved:

Theorem. *Let R be a Hodge algebra over K on H governed by Σ. Let h_0 be minimal in* $\mathrm{Ind}(R)$*. Then with H^*, Σ^* as above and the grading defined on* gr_I *above, so that* $\deg(u^*) = \deg(u)$ *when u is standard,* $\mathrm{gr}_I R$ *is a Hodge algebra over K on H^* governed by Σ^* (hence, with the same data as R). Moreover,*

$$\mathrm{Ind}(\mathrm{gr}_I R) \subseteq \{h^* : h \in \mathrm{Ind}(R)\} - \{h_0^*\}. \quad \square$$

We can now take the associated graded ring with respect to the ideal generated by a minimal element of the indiscrete part of the new algebra again, and do so repeatedly. Each repetition decreases the cardinality of the indiscrete part by at least one. We must eventually reach the discrete Hodge algebra over K on H governed by Σ.

Corollary. *Let R be a Hodge algebra over K on H governed by Σ. Then there is a finite sequence of Hodge algebras over K with the same data, each of which is the associated graded ring of its predecessor with respect to a principal homogenous ideal, and such that the last element in the sequence is, up to isomorphism, the discrete Hodge algebra over K on H governed by Σ.* $\square$

We shall embark on a program of getting information about a Hodge algebra from information about the corresponding discrete Hodge algebra. For the Cohen-Macaulay property, the following fact will be useful. Note that if (R, m, K) is local, $R(t)$ will denote the localization of the polynomial ring in one variable $R[t]$ at $mR[t]$, which is faithfully flat over R, with closed fiber $k(t)$.

Theorem. *Let $\mathcal{P}$ be a property of Noetherian rings such that:*
(1) *R has property $\mathcal{P}$ if and only if all local rings of R have property $\mathcal{P}$.*
(2) *If x is a nonzerodivisor in m, and R/xR has property $\mathcal{P}$, then R has property $\mathcal{P}$.*
(3) *If $R(t)$ has property $\mathcal{P}$ then R does.*

The Cohen-Macaulay property is an example.

Let I be an ideal of R such that $\mathrm{gr}_I(R)$ has property $\mathcal{P}$. Then R_Q has property $\mathcal{P}$ for every prime ideal Q of R such that $Q+I$ is a proper ideal of R, and, in particular, for every prime ideal of R that contains I.

Before giving the proof we recall that if I is an ideal of R and t is an indeterminate then the *Rees ring* of I is $R[It] \subseteq R[t]$ which may also be written as

$$R + It + I^2t^2 + I^3t^3 + \cdots .$$

Let $v = 1/t$ in $R[t]_t = R[t,\, 1/t]$. The *second Rees ring* of I is $R[It][v] \subseteq R[t,\, 1/t]$ which is easily seen to be:

$$\cdots + Rv^3 + Rv^2 + Rv + R + It + I^2t^2 + I^3t^3 + \cdots .$$

Then

$$vR[It,\, v] = \cdots + Rv^3 + Rv^2 + Rv + I + I^2t + I^3t^2 + I^4t^3 + \cdots ,$$

and we see that $R[It,\, v]/(v)$ may be identified with

$$R/I + (I/I^2)t + (I^2/I^3)t^2 + (I^3/I^4)t^3 + \cdots .$$

Here, the powers of t are just "place holders." This ring is $\mathrm{gr}_I R$. Moreover, since v is a unit in $R[t, 1/t]$, it is evidently a nonzerodivisor in this ring and in $R[It,\, v]$. We use v to denote $1/t$ here as a reminder that v is generally *not* a unit in $R[It,\, v]$, since t is not in the ring. We are now ready to give the proof of the theorem stated above.

Proof. If $Q+I$ is proper it is contained in a maximal ideal m of R. Since R_Q is a localization of R_m, it suffices to show that R_m has the required property. Since

$$\mathrm{gr}_{IR_m} R_m \cong \left(\mathrm{gr}_I R\right)_m ,$$

we may replace R by R_m. We henceforth assume that $(R,\, m,\, K)$ is local with $I \subseteq m$. Let

$$Q = \cdots + Rv^3 + Rv^2 + Rv + m + It + I^2t^2 + I^3t^3 + \cdots\,,$$

which is the kernel of the composite map

$$R[It,\, v] \twoheadrightarrow R[It,\, v]/(v) \cong \mathrm{gr}_I R \twoheadrightarrow R/I \twoheadrightarrow R/m.$$

Let

$$P = \cdots + mv^3 + mv^2 + mv + m + It + I^2t^2 + I^3t^3 + \cdots$$

which is the contraction of $mR[t,\, 1/t]$ to $R[It,\, v]$ and is the kernel of a surjection $R[It,\, v] \twoheadrightarrow K[v]$. Note that $P \subseteq Q$. Then v is a nonzerodivisor in the maximal ideal of $R[It,\, v]_Q$, and the quotient may be identified with $\big(R[It,\, v]/(v)\big)_Q \cong \mathrm{gr}_I(r)_Q$. Since this is a localization of $\mathrm{gr}_I(R)$, it has property $\mathcal{P}$, and then (2) implies that $R[It,\, v]_Q$ has property $\mathcal{P}$. Since $P \subseteq Q$, we see that $R[It,\, v]_P$ has property $\mathcal{P}$. In this ring, since $v \notin P$ it acquires an inverse in the localization, i.e., t is in the ring, and so this ring isomorphic with $R[t,\, v]_{P'}$, where P' denotes the expansion of P. But $P' = mR[t,\, v]$, and so $R[t,\, v]_{P'} \cong R(t)$. Since $R(t)$ has the property $\mathcal{P}$, R has the property $\mathcal{P}$. $\square$