Math 711: Lecture of September 30, 2005

We next note:

Proposition. *Let R be a ring and I an ideal of R. If $\bigcap_n I^n = 0$ (which is always true if R is $\mathbb{N}$-graded and $I \subseteq \bigoplus_{j=1}^{\infty} R_j$), then:*

(a) *If $\mathrm{gr}_I R$ is reduced, then R is reduced.*
(b) *If $\mathrm{gr}_I R$ is a domain, then R is a domain.*

Proof. For part (a), suppose $r \in R - \{0\}$ and $r^t = 0$. Choose j such that $r \in I^j - I^{j+1}$. Then $[r] \in I^j/I^{j+1}$ is a nonzero element of degree j in $\mathrm{gr}_I R$, and then $[r]^t$ is the image of r^t in I^{jt}/I^{jt+1}, and so is 0. Similarly, for part (b), if r, s are nonzero elements of R such that $rs = 0$ we can choose j such that $r \in I^j - I^{j+1}$ and $s \in I^k - I^{k+1}$. Then $[r] \in I^j/I^{j+1}$ and $[s] \in I^k/I^{k+1}$ are nonzero elements of degrees j and k respectively such that $[r][s]$ is 0 in I^{j+k}/I^{j+k+1}, since it is represented by $rs = 0$. $\square$

By the *socle* $\mathrm{Soc}(M)$ of a module M over a local ring (R, m, K) we mean $\mathrm{Ann}_M m \cong \mathrm{Hom}_R(K, M)$, which is a K-vector space.

Recall that a finitely generated module M over a local ring R is *Cohen-Macaulay* if $\mathrm{depth}_m M = \dim(M)$: we always have that $\mathrm{depth}_m M \leq \dim(M)$, where $\dim(M) = \dim(R/\mathrm{Ann}_R M)$. When R is Cohen-Macaulay over R, it is also Cohen-Macaulay over $R/\mathrm{Ann}_R M$, and the maximal regular sequences in m on M are the sequences of elements whose images in $R/\mathrm{Ann}_R M$ form a system of parameters for $R/\mathrm{Ann}_R M$. By the *type* of a Cohen-Macaulay module M of dimension d over a local ring (R, m, K) we mean the K-vector space dimension of the K-vector space $\mathrm{Ext}^d_R(K, M)$. If x is a nonzerodivisor in R on M, the short exact sequence

$$0 \to M \xrightarrow{x} M \to M/xM \to 0$$

yields a long exact sequence when we apply $\mathrm{Hom}_R(K, \_)$. In general, $\mathrm{Ext}^i_R(N, M)$ vanishes for $i < \mathrm{depth}_{\mathrm{Ann}_R N} M$, and so $\mathrm{Ext}^i_R(K, M)$ vanishes for $i < d$, while $\mathrm{Ext}^i_R(K, M/xM)$ vanishes for $i < d-1$. The first few nonzero terms in the long exact sequence for Ext have the form:

$$0 \to \mathrm{Ext}^{d-1}_R(K, M/xM) \to \mathrm{Ext}^d(K, M) \xrightarrow{x} \mathrm{Ext}^d(K, M).$$

Since $x \in \mathrm{Ann}_R K = m$, x kills $\mathrm{Ext}^d_R(K, M)$, and so we get an isomorphism

$$\mathrm{Ext}^{d-1}_R(K, M/xM) \cong \mathrm{Ext}^d(K, M)$$

It follows that the type of M is the same as the type of M/xM. Iterating, we find that if $x_1, \ldots, x_d$ is a maximal regular sequence on M, the type of M is the same as the type of $M/(x_1, \ldots, x_d)M$, which is

$$\dim_K\big(\mathrm{Hom}_R(K, M/(x_1, \ldots, x_d)M)\big) = \dim_K\big(\mathrm{Soc}(M/(x_1, \ldots, x_d)M\big)$$

The *type* of a Cohen-Macaulay local ring is simply its type as a module over itself. We define a local ring to be *Gorenstein* if it is Cohen-Macaulay of type 1. There are many other characterizations.

The next theorem will prove extremely valuable. We need a lemma first.

Lemma. *Let M be a flat R-module and let I be a finitely generated ideal of R. Then*

$$\mathrm{Ann}_M I = (\mathrm{Ann}_R I)M \cong (\mathrm{Ann}_R I) \otimes_R M.$$

Proof. Let $I = (f_1, \ldots, f_h)R$. Let A be the the $1 \times h$ matrix $(f_1\ f_2\ \cdots f_h)$. Then we have an exact sequence

$$0 \longrightarrow \mathrm{Ann}_R I \longrightarrow R \xrightarrow{A} R^h.$$

Since M is flat, applying $M \otimes_R \_$ yields an exact sequence

$$0 \longrightarrow (\mathrm{Ann}_R I) \otimes_R M \longrightarrow M \xrightarrow{\mathrm{id}_M \otimes A} M^h,$$

while the kernel of $\mathrm{id}_M \otimes_R A : M \to M^{\oplus h}$ is, evidently, $\mathrm{Ann}_M I$. The identification of $J \otimes M$, where $J = \mathrm{Ann}_R I$, with JM is a consequence of the injectivity of the map obtained from $0 \to J \subseteq R$ by applying $\_ \otimes_R M$: we have an injection $0 \to J \otimes_R M \hookrightarrow M$ whose image is JM. $\square$

We are now ready to prove:

Theorem. *Let $(R, m, K) \to (S, n, L)$ be a local homomorphism and let M be a finitely generarted S-module that is R-flat. (The most important case is when $M = S$ is R-flat.) Then:*

(a) $\dim(M) = \dim(R) + \dim(M/mM)$.

(b) $\mathrm{depth}_n M = \mathrm{depth}_m R + \mathrm{depth}_n M/mM$.

(c) *An element $x \in n$ is a nonzero divisor on M/mM if and only if it is a nonzerodivisor on M/IM for every ideal $I \subseteq m$ of R. Moreover, if x is a nonzerodivisor on M/mM then M/xM is R-flat.*

(d) *If $\mathrm{depth}_m R = 0$, then $x \in n$ is a nonzero divisor on M if and only if it is a nonzero divisor on M/mM.*

(e) *M Cohen-Macaulay if and only if R and M/mM are both Cohen-Macaulay.*

(f) *If M is Cohen-Macaulay, the type of M is the product of the types of R and M/mS.*

Proof. We observe for parts (a) and (b) that when we replace R by R/J, S by S/JS, and M by $(R/J) \otimes_R M \cong M/JM$, we still have that M/JM is flat over R/J and the closed fiber of M over R, namely M/mM, is not affected. For parts (a) and (b) we use Noetherian induction, and assume that R has been replaced by R/J, where J is maximal with respect to the property of giving a counter-example. Thus, we may assume that the result holds when R is replaced by any proper quotient.

For part (a), we note first that R must be reduced, for if we take J to be the nilradical of R, R/J and M/JM have the same dimensions as R and M, while M/mM does not change when we replace R by R/J. Therefore we may assume that R is reduced. If $\dim(R) = 0$ then $R = K$, $m = 0$, $M/mM = M$, and the result is obvious.

If R has positive dimension then, since R is reduced, we can choose a nonzerodivisor $x \in m$. Since M is R-flat, x is also a nonzerodivisor on M, so that $\dim(R/xR) = \dim(R) - 1$ and $\dim(M/xM) = \dim(M) - 1$. Since the result holds for R/xR, S/xS, and M/xM by the induction hypothesis, we have that

$$\dim(M) - 1 = \dim(M/xM) = \dim(R/xR) + \dim(M/mM) = \dim(R) - 1 - \dim(M/mM),$$

and the required result follows by adding 1 to both sides.

Before proving (b), we prove (c) and then (d). To prove the statement in (c) we assume that $x \in n$ is a nonzerodivisor on M/m. We replace R byR/J where J is maximal with respect to giving a counter-example, and so we may assume the result holds for every ideal of R except possibly 0. If $\mathrm{depth}_R m = 0$, we let J be uR, where u is a nonzero element of m killed by m. Then R has a filtration with the factors $J \cong K$ and R/J, and so M has a filration with the factors $J \otimes_R M$ and $(R/J) \otimes_R M = M/JM$. The former is $K \otimes_R M = M/mM$, and x is not a zerodivisor on this module by hypothesis. But x is also not a zerodivisor on M/JM by the hypothesis of Noetherian induction, and so x is not a zerodivisor on M.

If $\mathrm{depth}_m R > 0$, we can choose y in m such that y is not a zerodivisor in R. Since M is R-flat, y is not a zerodivisor on M. By the hypothesis of Noetherian induction, x is not a zerodivisor on M/yM. But then y, x is a regular sequence in S on M. Since S is local and M is finitely generated over S, regular sequences are permutable, and x, y is a regular sequence on M. But this shows that x is not a zerodivisor on M.

This establishes all but the last statement in part (c). To prove that M/xM is again R-flat, it suffices to prove that $\mathrm{Tor}_1^R(R/I, M/xM) = 0$ for every ideal I of R. Apply $(R/I) \otimes_R \_$ to the short exact sequence $0 \to M \to M \to M/xM \to 0$, where $M \to M$ is multiplication by x. Then the long exact sequence for Tor yields an exact sequence:

$$\mathrm{Tor}_1^R(R/I, M) \to \mathrm{Tor}_1^R(R/I, M/xM) \to M/IM \to M/IM$$

where the map $M/IM \to M/IM$ is multiplication by x and so is injective by what we have already shown. Since M is R-flat, $\mathrm{Tor}_1^R(R/I, M) = 0$, and so $\mathrm{Tor}_1(R/I, M/xM) = 0$, as required.

This completes the proof of (c). For part (d), it suffices to show that if R has depth 0, and x is not a zerodivisor on M then x is not a zerodivisor on M/mM: the converse, in a much stronger form, has already been proved in part (c). The fact that $\mathrm{depth}_m R = 0$ implies that there is an embedding $K \hookrightarrow R$. Applying $M \otimes_R \_$ yields an embedding of $M/mM \hookrightarrow M$. The fact that x is a nonzerodivisor on M implies that is a nonzerodivisor on every submodule of M, and, hence, on M/mM, as required.

We can now prove (b). We use inductnion on $\mathrm{depth}(R)+\mathrm{depth}(M/mM)$. If $\mathrm{depth}_m R > 0$ then we can choos $x \in m$ that is a nonzerodivisor in R and, hence, on M. We pass to R/xR, S/xS, and M/xM. The depths of R and M decrease by 1, while M/mM stays the same. Then

$$\mathrm{depth}_n M - 1 = \mathrm{depth}_n M/xM = \mathrm{depth}_m R/xR + \mathrm{depth}_n M/mM$$

by the induction hypothesis. The latter is $\text{depth}_m R - 1 + \text{depth}_n M/xM$, and the result follows.

Now assume that $\text{depth}_m R = 0$. By part (d), $\text{depth}_n M > 0$ iff $\text{depth}_n M/mM > 0$. If both are 0, the result is clear. If both are positive, then we can choose $x \in n$ that is a nonzerodivisor on M/mM, and then x is a nonzerodivisor on M as well, by part (c). By part (c), M/xM is R-flat. We can apply the induction hypothesis to R, S, and M/xM. The new closed fiber is $(M/xM)/m(M/xM) \cong (M/mM)/x(M/mM)$, and since x is a nonzerodivisor on M/mM, we have

$$\text{depth}_n M - 1 = \text{depth}_n M/xM = \text{depth}_m R + \text{depth}_n (M/mM)/x(M/mM)$$

by the induction hypothesis. The latter is $\text{depth}_m R + \text{depth}_n M - 1$, and the result follows.

To prove (e) we note that we have, general:

$$\text{depth}_m R \leq \dim(R)$$

and

$$\text{depth}_n M/mM \leq \dim(M/mM).$$

If both are equalities we may add and apply parts (a) and (b) to get that $\text{depth}_n M = \dim(M)$. If either inequality is strict we may add and apply parts (a) and (b) to get the strict inequality $\text{depth})_n M < \dim(M)$.

For part (f), assume that M is Cohen-Macaulay, so that both R and M/mM are Cohen-Macaulay. Let $x_1, \ldots, x_d$ be a maximal regular sequence in R, which is a regular sequence on M. We pass to $R/(x_1, \ldots, x_d)R$, $S/(x_1, \ldots, x_d)S$, and $M/(x_1, \ldots, x_d)M$. The types of R and M don't change, and M/mM does not change. Therefore we may assume that R is an Artin local ring. If M/mM is not zero-dimensional we can choose $x \in n$ not a zerodivisor on it, and pass to R, S/xS, M/xM. The type of M does not change. Iterating, we see that we may assume that $\dim(M/mM) = 0$.

The annihilator of n in M is contained in $\text{Ann}_M m$, which, by the Lemma, may be identified with $(\text{Ann}_R m) \otimes_R M$. But $\text{Ann}_R m \cong K^t$ is the type of R. Thus, $\text{Ann}_M n$ may be identified with $\text{Ann}_{K^t \otimes_R M} n$, and $K^t \otimes_R M \cong (M/mM)^{\oplus t}$, and so we need only determine the annihilator of n in this module. Since the annihilator of N in M/mM is isomorphic with $L^{t'}$, where t' is the type of M/mM, we obtain $(L^{t'}) \oplus t \cong L^{tt'}$, as required. $\square$

Corollary. *If $(R, m, K) \to (S, n, L)$ is a flat local homormorphism then S is Cohen-Macaulay if and only if both R and S/mS are Cohen-Macaulay, and S is Gorenstein if and only if both R and S/mS are Gorenstein.*

Proof. The first statement is a special case of part (e) of the preceding theorem, and the second statement follows from part (f) of the preceding theorem, since the type of S is the product of the types of R and S/mS when S is Cohen-Macaulay. $\square$