Math 711: Lecture of October 3, 2005

Let S be an additive semigroup with identity 0 (we are assuming that the semigroup operation is associative). A ring R is *S-graded* if it has a direct sum decomposition

$$R = \bigoplus_{s \in S} R_s$$

such that $1 \in R_0$ and for all $s,\, t \in S$, $R_s R_t \subseteq R_{s+t}$. If R is S-graded an R-module M is said to be S-graded if it has a direct sum decomposition $M = \bigoplus_{s \in S} M_s$ such that for all $s,\, t \in S$, $R_s M_t \subseteq M_{s+t}$. An element of some R_s or M_s is said to be *homogeneous* of degree s or a *form* of degree s. A submodule of $N \subseteq M$, where M is graded, is called *graded* or *homogenous* if

$$N = \bigoplus_{s \in S} (N \cap M_s),$$

or if, equivalently, N is generated by homogeneous elements.

S is said to be a *a cancellation semigroup* if whenever $s+u = t+u$ for $s,\, t,\, u \in S$, one has that $s = t$. S is said to have a linear order *compatible with addition* if it has a linear order $\leq$ such that for all $s,\, t,\, u \in S$, if $s \leq t$ then $s+u \leq t+u$. Note that if $s \leq t$ and $u \leq v$, we have that $s+u \leq t+v$, since $s+u \leq t+u \leq t+v$. Every subsemigroup of a cancellation semigroup is a cancellation semigroup, and every subsemigroup of a semigroup with a linear order compatible with its addition has a linear order compatible with its addition: one simply restricts the linear order on the larger semigroup. In particular, every subsemigroup of $\mathbb{Z}^n$ is a cancellation semigroup with a linear order compatible with addition, since this is true of $\mathbb{Z}^n$: cancellation holds because $\mathbb{Z}^n$ is a group, and for the linear order we may define $(a_1,\, \ldots\, , a_n) \leq (b_1,\, \ldots\, , b_n)$ if the two are equal or if there exists i, $1 \leq i \leq n$, such that $a_i = b_i$ for $i < j$ while $a_j < b_j$.

Proposition. *Let R be a Noetherian ring graded by a semigroup S such that S has cancellation and also has a linear order compatible with addition. Let M be an S-graded R-module. Then any associated prime of M (i.e., any prime of R that is the annihilator of an element of M) is homogeneous. Hence, the nilradical* $\mathrm{Rad}\,(0)$ *of R is homogeneous. More generally, the radical of a homogeneous ideal $\mathfrak{A}$ is homogeneous.*

Proof. The final statement follows from the next to last statement applied to $R/\mathfrak{A}$, while the next to last statement follows from the first statement because it implies that any minimal prime of R is homogeneous, and an intersection of homogeneous ideals is homogeneous. To prove the first statement let $u \in M - \{0\}$ have prime annihilator P. Then every nonzero multiple of u has annihilator equal to P, for if $ru \neq 0$ then $Pru = 0$, while if $r' \notin P$, $rr'u = 0$ implies that $rr' \in P$ and so $r \in P$, which implies that $ru = 0$, a contradiction.

Among all those elements u having P as annihilator, choose u so that the number k of its nonzero homogeneous components is minimum. Then we can write $u = u_{s_1} + \cdots + u_{s_k}$, where $u_{s_j} \in M_{s_j}$ for every j, $s_1 < \cdots < s_k$, every $u_{s_j} \neq 0$, and k is minimum.

We shall show that all of the ideals $\mathrm{Ann}_R u_{s_j}$ are homogeneous, and that they are all equal. Note that if v is a homogeneous element then v kills a sum of forms of distinct degrees if and only if v kills each of them: this uses the fact that S is a cancellation semigroup. It follows that the annihilator of a homegeneous element is a homogeneous ideal.

Now suppose that v is a homogeneous element of one of the ideals $\mathrm{Ann}_R u_{s_j}$. If v does not kill u_{s_i} for some i, then vu will be a nonzero multiple of u with fewer nonzero homogeneous components than u, a contradiction. This shows that all of the homogeneous ideals $\mathrm{Ann}_R u_{s_j}$ are the same. It follows that any element in $\mathrm{Ann}_R u_{s_k}$ kills u and so is in P.

We shall show that all of these are the same as P. Consider any element $r = r_{t_1} + \cdots + r_{t_h}$ of $\mathrm{Ann}_R u$, where $t_1 < \cdots < t_h$ and each $r_{t_\nu} \in R_{t_\nu}$. We shall prove that every r_{t_ν} kills u_{s_k}. This will show that $r \in \mathrm{Ann}_R u_{s_k}$, and so $P \subseteq \mathrm{Ann}_R u_{s_k}$. The point is that when one expands ru by the distributive law, there is a unique highest degree term $r_{t_h} u_{s_k}$, and so $r_{t_h} u_{s_k} = 0$. Since $r u_{s_k} = 0$, we can conclude that $r_{t_1} + \cdots + r_{t_{h-1}}$ also kills u, and the fact that all the r_{t_ν} kill u_{s_k} now follows by induction on h. $\square$

Example. Some hypothesis on S is needed here. E.g., we may grade the polynomial ring $\mathbb{Z}_2[x,\, y]$ with $S = \mathbb{Z}_2$ by letting x have degree 0 and y have degree 1. The degree if the monomial $x^a y^b$ is 0 if b is even and 1 if b is odd. In this ring, $(x+y)^2 = x^2 + y^2$ is homogeneous, but $x+y$ is not. Thus, the radical of the homogeneous ideal (x^2+y^2) is not homogeneous. The problem is that while $\mathbb{Z}_2$, a group, has cancellation, it does not have a linear order compatible with addition.

The polynomial ring $R = K[x_1,\, \ldots\,, x_n]$ over any base ring K has an $\mathbb{N}^n$ grading, also referred to as *the grading by monomials*, such that $R_{(a_1,\, \ldots\,, a_n)} = K x_1^{a_1} \cdots x_n^{a_n}$. The degree of x_i is the ith standard basis vector e_i for $\mathbb{N}^n$. The graded ideals with respect to this grading of R are precisely the ideals generated by monomials.

Corollary. *Let K be a field and Δ a finite simplicial complex with vertices $x_1, \ldots, x_n$. Then $K[\Delta]$ is reduced.*

Proof. Since $K[\Delta] = K[x_1,\, \ldots\,, x_n]/I_\Delta$, it suffices to show that I_Δ is radical. Since the radical of I_Δ is homogenous, it suffices to show that if a monomial u has the property that $u^t \in I_\Delta$, $t \geq 1$, then $u \in I_\Delta$. But a monomial is in I_Δ if and only if the set of variables occurring with a positive exponent (the support) is not a face of Δ, and this set is the same for u^t and u. $\square$

Coupled with our earlier work, this tells us the primary decomposition of I_Δ: it is the intersection of its minimal primes, which are in bijective correspondence with the facets of Δ: each is generated by the set of variables that is the complement of some facet.

Corollary. *Let R be a Hodge algebra over K on H governed by Σ, and suppose that K is a field and each generator of Σ takes on only the values 0 and 1 on H. Then R is reduced.*

Proof. There is a sequence of successive associated graded rings that begins with R and ends with the corresponding discrete Hodge algebra on K with the same data. The condition on the generators of Σ implies that the discrete Hodge algebra is a quotient of the polynomial ring on variables corresponding to the elements of H by an ideal that is generated by square-free monomials, and so the discrete Hodge algebra is a face ring. It is therefore reduced. By induction on the number of associated graded algebras in the chain, we see that each of them is reduced. $\square$

Corollary. *If R is an ASL over the field K then R is reduced.*

Proof. In this case Σ is generated by functions on H that take the value 1 on two incomparable elements of H and are 0 elsewhere. $\square$

Given a poset H, there is an associated simplicial complex whose vertices are the elements of H: it consists of all subsets of H that are linearly ordered. This simplicial complex is called the *order complex* of H. The discrete Hodge algebra associated with an ASL R over K is the face ring over K of the order complex of H.

We want to make similar deductions for the Cohen-Macaulay and Gorenstein properties. In order to do so, we need to study the Gorenstein property further, and to do so we use the notion of Ext-duals of Cohen-Macaulay modules over regular rings.

We begin by reviewing some properties of Cohen-Macaulay modules.

We first consider the case where the ring is local.

Proposition. *Let (R, m, K) be a local ring and let M be a finitely generated R-module with Krull dimension d and annihilator I. The following conditions are equivalent:*

(1) *M is Cohen-Macaulay.*
(2) $\mathrm{depth}_m M = d$.
(3) *M is Cohen-Macaulay as an R/I-module.*
(4) *Some system of parameters for R/I is a regular sequence on M.*
(5) *Every system of parameters for R/I is a regular sequence on M.*
(6) *A sequence of elements of R is a regular sequence on M if and only if its image in R/I it is part of a system of parameters for R/I.*
(7) *For every ideal $J \subseteq M$, the depth of M on J is the same as the height of $J(R/I)$ in R/I.*

If these equivalent conditions hold, then we also have:

(a) *For every prime ideal P of R, M_P is Cohen-Macaulay over R_P.*
(b) *If $x_1, \ldots, x_k \in m$ are such that their images in R/I are part of a system of parameters, then $M/(x_1, \ldots, x_k)M$ is Cohen-Macaulay of dimension $d - k$.*
(c) *Every nonzero submodule of M has dimension d. Thus, if P is an associated prime of M, then R/P has dimenson d. Consequently, M has no embedded primes, and the associated primes of M are the same as the minimal primes of I.*

If R is regular, M is Cohen-Macaulay and if and only if $\mathrm{pd}_R M = depth_I R$ (which is the same as the height of I, since R is Cohen-Macaulay). Moreover, in this case, if h is the height of I, $\mathrm{Ext}^i_R(M, R)$ vanishes except when $i = h$.

Proof. (2) is the definition of Cohen-Macaulay, and neither the depth nor dimension of M is affected when we replace R by R/I. Thus, the first three conditions are equivalent. In proving the remaining equivalences we replace R by R/I, so that $\dim(R) = \dim(M) = d$. The depth of M on an ideal only depends on the radical. We now make use of the results relating depth and Koszul homology from the Lecture Notes from February 18 from Math 615, Fall 2004. If $x_1, \ldots, x_d$ is one system of parameters then $\mathrm{depth}_m M = d$ if and only if $\mathrm{depth}_{(x_1, \ldots, x_d)} M = d$ if and only if all of the higher Koszul homology of $\mathcal{K}_\bullet(x_1, \ldots, x_d;\, M)$ vanishes, and this holds for a finitely generated module over a local ring iff $x_1, \ldots, x_d$ is a regular sequence on m. Thus, (4) $\Rightarrow$ (3) $\Rightarrow$ (5) $\Rightarrow$ (4) is clear. To see that (6) is equivalent to the others, first note that since any regular sequence can be extended to a maximal regular sequence of length d, all we need to show is that a maximal regular sequence on M must consist of parameters for R. We know that if x is a nonzerodivisor on M, then $\dim(M/xM) = \dim(M) - 1$. Hence, with $N = R/(x_1, \ldots, x_d)R$, we have that $\dim(M \otimes_R N) = \dim\big(M/(x_1, \ldots, x_d)M\big) = \dim(M) - d = 0$, which shows that $M \otimes_R N$ has finite length and therefore is supported only at m. Now $\mathrm{Supp}(M \otimes_R N) = \mathrm{Supp}(M) \cap \mathrm{Supp}(N) = \mathrm{Spec}(R) \cap \mathrm{Supp}(N) = \mathrm{Supp}(N)$, and the fact that $\mathrm{Supp}\big(R/(x_1, \ldots, x_d)R\big) = \{m\}$ implies that $\mathrm{Rad}(x_1, \ldots, x_d)R = m$, which in turn yields that $x_1, \ldots, x_d$ is a system of parameters for R. (Keep in mind that we have passed to the case where $I = 0$, i.e., where M is faithful over R.)

It is clear that (7) $\Rightarrow$ (2), since we may take $J = m$. We postpone the proof of the converse implication until we have proved statement (c) about Cohen-Macaulay modules.

To prove (c), suppose that M is Cohen-Macaulay but has a nonzero submodule N of lower dimension. Then it has a maximal such submodule. Since the sum of two such submodules has the same property, we may assume that N is maximum. If $di(M) = 0$ there result is obvious, and so we may assume that $\dim(M) > 0$. It is clear that M/N has no submodule of dimension smaller than M or we could enlarge N. In particular, m is not an associated prime of M/N, and we can therefore choose $x \in M$ so as to avoid the associated primes of M/N and the minimal primes of R. It follows that x is a nonzero divisor on both M and M/N. The short exact sequence

$$0 \to N \to M \to M/N \to 0$$

therefore remains exact when we apply $R/xR \otimes_R \_$, and so we have an embedding $N/xN \hookrightarrow M/xM$. But $\dim(N/xN) = \dim(N) - 1 < \dim(M) - 1 = \dim(M/xM)$, and this produces a counter-example in the Cohen-Macaulay module M/xM of lower dimension than M. Note here that with $N \neq 0$, we have $N/xN \neq 0$ by Nakayama's Lemma. The statement about associated primes follows because P is associated if and only if R/P embeds in M. Thus, the associated primes of M are the same as the minimal primes of M, and these minimal primes in $\mathrm{Supp}(M) = V(I)$ are the same as the minimal primes of I.

Thus, the minimal primes of R/I all have quotients of dimenson $d = \dim(R/I)$. With this hypothesis on R/I, we claim that the height of $J(R/I)$ is the same as the length of the longest sequence $x_1, \ldots, x_k$ in $J(R/I)$ that is part of a system of parameters. We leave this statement as an exercise. The equivalence of the first six conditions with (7) is now immediate.

To prove (a), again replace R by R/I and P by P/I. Suppose that P has height h. Then, by the preceding paragraph, there is part of a system of parameters $x_1, \ldots, x_h$ in P, and these elements form a regular sequence on M. Thus, there images in R_P form a regular sequence of length h on M_P, and $\mathrm{depth}_{PR_P} M_P \geq h = \dim(R_P)$. The other inequality always holds, and so M_P is Cohen-Macaulay over R_P.

Part (b) is clear.

Now consider the case where R is regular. Then M is Cohen-Macaulay if and only if

$$\mathrm{depth}_m M = \dim(M) = \dim(R/I) = \dim(R) - \mathrm{height}(I).$$

Since

$$\mathrm{pd}_R M = \mathrm{depth}(R) - \mathrm{depth}_m(M) = \dim(R) - \mathrm{depth}_m(M),$$

the Cohen-Macaulay condition becomes that

$$\dim(R) - \mathrm{pd}_R M = \dim(M) = \dim(R) - \mathrm{height}(I),$$

and this is eqivalent to the condition that $\mathrm{pd}_R M = \mathrm{height}(I)$. Note that since R is regular, R is Cohen-Macaulay, and $\mathrm{height}(I) = \mathrm{depth}_I R$.

Finally, $\mathrm{Ext}^i_R(M, R)$ vanishes for $i > \mathrm{pd}_R M = h$, and also for $i < \mathrm{depth}_{\mathrm{Ann}_R M} R = \mathrm{depth}_I R = h$ as well. $\square$

If R is Noetherian, a finitely generated R-module MN is called *Cohen-Macaulay* if M_P is Cohen-Macaulay over R_P for every prime (equivalently, for every maximal) ideal P of R. Evidently, if W is a multiplicative system in R, then $W^{-1}M$ is Cohen-Macaulay over $W^{-1}R$. Note that if the annihilator of M is height unmixed and $W^{-1}M \neq 0$, then the annihilator of $W^{-1}M$ is height unmixed in $W^{-1}R$: one gets the expansions of the minimal primes of M that do not meet W.

When R is regular, there is still a strong tendency for $I = \mathrm{Ann}_R M$ to be height unmixed, i.e., for all minimal primes to have the same height. Specifically:

Proposition. *Let R be a regular ring and let M be a Cohen-Macaulay module over R with annihilator I. If* $\mathrm{Supp}(M)$*, which is $V(I)$, is connected, then I is height unmixed, i.e., all minimal primes of I have the same height.*

Proof. Suppose that the minimal primes of I have more than one height. Let $P_1, \ldots, P_s$ be the minimal primes of height h, and let $Q_1, \ldots, Q_t$ be the minimal primes of other height. Let $J = \bigcap_{i=1}^s P_i$ and $J' = \bigcap_{j=1}^t Q_j$. Then $V(I) = V(J \cap J') = V(J) \cup V(J')$, and $V(J)$, $V(J')$ are disjoint: if a prime m were in both, M_m, a Cohen-Macaulay module over R_m, would have minimal primes $P_i R_m$ and $Q_j R_m$ of different heights, and we already know that this is impossible in the local case. This shows that $\mathrm{Supp}(M)$ is not connected. $\square$

Now let M be a Cohen-Macaulay module over a regular ring R such that $\mathrm{Ann}_R M$ has pure height h. For such a module M, we define the *Ext dual* M^* of M as $\mathrm{Ext}^h_R(M, R)$. Evidently, M^* is a finitely generated R-module. We shall see that it is Cohen-Macaulay, with the same annihilator as M, and that its dual is M.

The following two results summarize many of the important properties of the Ext dual.

Theorem. *Let R be a regular Noetherian ring and let M, M', M'' be Cohen-Macaulay modules with annihilators of pure height h. Let N be Cohen-Macaulay with annihilator of pure height $h+1$.*

(a) *M^* is Cohen-Macaulay with the same annihilator as M.*
(b) *There is a natural isomorphism $M \to M^{**}$. Thus, up to isomorphism, each of M and M^* is the dual of the other.*
(c) *If $0 \to M' \to M \to M'' \to 0$ is exact then $0 \to M''^* \to M^* \to M'^* \to 0$ is exact.*
(d) *$0 \to M' \to M \to N \to 0$ is exact then $0 \to M^* \to M'^* \to N^* \to 0$ is exact.*
(e) *If x is a nonzerodivisor on M and $xM \neq M$, then M/xM is Cohen-Macaulay with annihilator of pure height $h+1$, x is not a zerodivisor on M^*, and $(M/xM)^* \cong M^*/xM^*$.*
(f) *If W is a multiplicative system in R and $W^{-1}M \neq 0$, then it is Cohen-Macaulay of pure height h and $(W^{-1}M)^* \cong W^{-1}(M^*)$.*

Proof. Part (f) follows from the fact that localization commutes with Ext when the first module is finitely presented.

We shall next prove (b). Note that when one localizes at any prime M, if M is not killed, it becomes Cohen-Macaulay with annihilator of pure height h. Thus, locally, M is either 0 or of projective dimension h. It follows that $\mathrm{pd}_R M = h$. Consider any projecitve resoluton

$$0 \to G_h \to \cdots \to G_0 \to 0$$

of M of shortest possible length h, where the G_i are finitely generated projective modules. The higher homology of $G_\bullet$ is 0, while $H_0(G_\bullet) = M$.

Note that for finitely generated projective modules G, the natural map $G \to G^{**}$ that sends u to the map whose value on $f \in G^*$ is $f(u)$ is an isomorphism, i.e., G is reflexive. (Here, G^* is $\mathrm{Hom}_R(G, R)$, but in this case that agrees with the Ext dual. R is easily checked to be reflexive, and the direct sum of two modules is reflexive if and only if both are. Thus, finite rank free modules are reflexive, and, hence, so are their direct summands. Moreover, the dual of a finitely generated free module is free, and so the dual of a finitely generated projective module is projective.) As in the local case, $\mathrm{Ext}^i_R(M, R) = 0$ except when $i = h$: this is obviously true when we localize at primes that do not contain I, and it is true if $P \supseteq I$ by the local results. The complex

$$0 \to G_0^* \to \cdots \to G_h^* \to 0,$$

with the numbering reversed, is therefore acyclic (the higher homology is $\mathrm{Ext}^i_R(M, R)$ for $i < h$) with augmentation M^*. It is now clear that if we use this projective resolution to calculate M^{**}, then we obtain the augmentation of the complex which is the double dual into R of $G_\bullet$: there is a natural isomorphism from $G_\bullet$ to its double dual. This induces an isomorphism $M \cong M^{**}$.

This is independent of the choice of length h projective resolution of M. Given two such, there are maps between them such that the composition in either direction is homotopic to the identity. Dualizing into R provides maps of the dual complexes with the same property. It is easy to check that the usual identifications of (first M^* and then)

M^{**} computed from the two different projective resolutions are compatible with the two identifications of M with M^{**}.

Since $\mathrm{Ann}_R M$ kills M^*, $\mathrm{Ann}_R M \subseteq \mathrm{Ann}_R M^*$. Similarly, $\mathrm{Ann}_R M^* \subseteq \mathrm{Ann}_R M^{**} = \mathrm{Ann}_R M$.

To prove that M^* is Cohen-Macaulay, we may pass to the local case of a local ring (R, m, K), by (f). Since M and M^* have the same annihilator, they have the same dimension. The resolution $G_\bullet^*$ exhibited for M^* shows that $\mathrm{pd}_R M^* \leq h$. Subtracting from $\dim(R)$ shows that $\mathrm{depth}_m M^* \geq \mathrm{depth}_m M = \dim(M) = \dim(M^*)$, and the other inequality always holds.

Parts (c) and (d) are immediate from the long exact sequence for Ext and the fact that a Cohen-Macaulay module with annihilator of pure height h has a unique non-vanishing Ext into R. Part (e) is then a special case of (d) once we show that the annihilator of M/xM has pure height $h+1$. Localize at a minimal prime P of the annihilator. The annihilator of M_P still has pure height h, and we are now in the case of a regular local ring, where the result is obvious. $\square$