Math 711: Lecture of October 5, 2005

We shall denote by $\nu(M)$ the least number of generators of a finitely generated R-module M. By Nakayama's Lemma, if (R, m, K) is local (or even quasilocal), $\nu(M) = \dim_K(M/mM)$.

Theorem. *Let (R, m, K) be a regular local ring of dimension n.*

(a) *$K^* \cong K$. For every finite length R-module M, M and M^* have the same length. (M is automatically Cohen-Macaulay.) Moreover, $\nu(M^*)$ is $\dim_K\bigl(\mathrm{Soc}(M)\bigr)$, which is the type of M. Of course, since $M^{**} \cong M$, we also have that $\nu(M) = \dim\bigl(\mathrm{Soc}(M^*)\bigr)$.*

(b) *Let M be a finitely generated Cohen-Macaulay module of dimension d. Then $\nu(M^*)$ is the type of M.*

Proof. For part (a), let $x_1, \ldots, x_n$ be minimal generators of R: they are also a regular sequence. Then $K^* = \mathrm{Ext}^n(K, R) \cong \mathrm{Hom}_R(K, R/(x_1, \ldots, x_n)R) = \mathrm{Hom}_R(K, K) \cong K$, as required. The statement that the lengths of M^* and M are equal is then immediate by induction: if M has length 1, then $M \cong K$ and we have already done this case. Otherwise, there is a short exact sequence $0 \to N \to M \to Q \to 0$ where N is a proper nonzero submodule of M, and then the length of M is the sum of the lengths of N and Q, which are both nonzero and, hence, both less than the length of M. We have a short exact sequence $0 \to Q^* \to M^* \to N^* \to 0$, and so the length of M^* is the sum of the lengths of Q^* and N^*. By the induction hypothesis, these are the same as the lengths of Q and N, which add up to the length of M, and we are done.

For the remaining statement note that we have a short exact sequence

$$0 \to \mathrm{Ann}_M m \to M \xrightarrow{\phi} M^{\oplus n}$$

where the map ϕ sends $u \mapsto (x_1 u, \ldots, x_n u)$. It is clear that the kernel of ϕ is $\mathrm{Ann}_M m$. Since the functor $\_^*$ is contravariant and exact on zero-dimensional modules, we obtain an exact sequence:

$$0 \leftarrow (\mathrm{Ann}_M m)^* \leftarrow M^* \xleftarrow{\phi^*} {M^*}^{\oplus n}$$

where it is easy to see that ϕ^* sends $(v_1, \ldots, v_n) \mapsto \sum_{j=1}^n x_n v_n$. Thus, it is clear that $\mathrm{Coker}\,(\phi^*) \cong (\mathrm{Ann}_M m)^*$. But the cokernel of ϕ^* is evidently M^*/mM^*, and since $(\mathrm{Ann}_M m)^*$ has the same K-vector space dimension as $\mathrm{Ann}_M m$, the result follows.

For part (b) note that if $x \in M$ is a nonzerodivisor on M, then M/xM has the same minimum number of generators as M, and its type is also the same as the type of M, while $(M/xM)^* \cong M^*/xM^*$, and so the minimal number of generators and the type of $(M/xM)^*$ are also unaffected. By iterating, we reduce to the case where M has finite length, which we settled in part (a). $\square$

Theorem. *Let M be a finitely generated Coohen-Macaulay module over a local (R, m, K), and let P be a prime ideal of R. Then the type of M_P is less than or equal to the type of M.*

Proof. We first consider the case where R is a homomorphic image of a regular local ring S. We then replace R by S and m and P by their inverse images in S. Thus, we may assume without loss of generality that R is regular. Then the type of M_P is the least number of generators of $(M_P)^* \cong (M^*)_P$, and this is evidently at most the number of generators of M^*, which is the type of M.

In the general case we consider the completion $\widehat{R}$ of R. Let Q be a minimal prime of $P\widehat{R}$ lying over P in R. The type of M is the same as the type of $\widehat{M}$ over $\widehat{R}$: a system of parameters for R is also one in $\widehat{R}$, and the quotients will be isomorphic. Since $\widehat{R}$ is a homomorphic image of a regular local ring, we have that the type of M equals the type of $\widehat{M}$, and is greater than or equal to the type of $\widehat{M}_Q$. It therefore suffices to show that the type of $\widehat{M}_Q$ is at least as large as the type of M_P. The following lemma completes the proof, with $B = R_P$, $M = M_P$, and $C = \widehat{R}_Q$.

Lemma. *Let M be a Cohen-Macaulay module over a local ring (B, m_B, K), and let $B \to C$ be a flat local homomorphism such that C/m_BC is zero-dimensional. Then $C \otimes M$ is Cohen-Macaulay over C, and its type is bigger than or equal to the type of M.*

Proof. Let $x_1, \ldots, x_d$ be a system of parameters for B. Then it is also a system of parameters for C. We replace B, C and M with tensor products over B with $B/(d_1, \ldots, d_)B$, and so assume that B and C both have dimension 0. If t is the type of M, then K^t embeds in M as $\mathrm{Ann}_M m_B$. Applying $C \otimes_B \_$ yields the direct sum of t copies of C/m_BC as a submodule of $C \otimes_B M$, which shows that the dimension of the socle in $C \otimes_B M$ over C is at least the product of the type of M and the type of C/m_BC. $\square$

The result below is true under various other hypotheses on R, e.g., if R is excellent or a homomorphic image of Cohen-Macaulay ring. We shall not not need such great generality here.

Theorem. *Let R be a Noetherian ring that is a homomorphic image of a regular ring. Let M be a finitely generated R-module. The set $\{P \in \mathrm{Spec}\,(R) : M_P \text{ is Cohen-Macaulay}\}$ is Zariski open in* $\mathrm{Spec}\,(R)$.

Proof. We may replace R by the regular ring that maps onto it without affecting the issue. Let $I = \mathrm{Ann}_R M$. After localizing at P, IR_P has pure height h. We want to show that we can choose $a \in R - P$ such that M_a is Cohen-Macaulay, and we are free to localize at one element of $R - P$ finitely many times to achieve this. We do not change notation as we localize. First, choose $a \notin P$ but in all minimal primes of P that do not have height h. After replacing R, M by R_a, M_a we may assume that I has pure height h. Since R is regular, M has finite projective dimension s. Consider the modules $\mathrm{Ext}^i_R(M, R)$ for $0 \le i \le s$ with $i \neq h$. When we localize at P, these finitely generated modules all become 0, and so there is a single element $a' \notin P$ that kills them all. Replace R, M by $R_{a'}$, $M_{a'}$, we may assume that $\mathrm{Ext}^i_R(M, R)$ vanishes except when $i = h$. This implies that M is Cohen-Macaulay. To see this, we may assume that we have localized at a single prime containing I. Call the local ring obtained (R, m, K). The vanishing of $\mathrm{Ext}^i_R(M, R)$ for $i > h$ shows that $\mathrm{pd}_R M \le h$ by the Lemma below, and so $\mathrm{depth}_m M \ge \dim(R) - h = \dim(\mathrm{R/I}) = \dim(\mathrm{M})$. The other inequality always holds. $\square$

Lemma. *Let (R, m, K) be a local ring and M a finitely generated nonzero module of finite projective dimension d. Then* $\mathrm{Ext}^d_R(M, R) \neq 0$ *(while, of course,* $\mathrm{Ext}^i_R(M, R) = 0$ *for $i > d$).*

Proof. Consider the last map of nonzero modules $f : R^{b_d} \to R^{b_{d-1}}$ in a minimal free resolution of M. If we use this resolution to compute $\mathrm{Ext}^\bullet_R(M, R)$ we see that $\mathrm{Ext}^d_R(M, R)$ is the cokernel of the map dual to f: the matrix of this map is the transpose of the matrix of f, and so the matrix has entries in m. It follows that the cokernel is nonzero. $\square$