## Math 711: Lecture of October 7, 2005

For the purpose of the next theorem, we make the convention that the type of the 0 module over a local ring $R$ is $\leq 1$. (It should be the vector space dimension of $\mathrm{Ext}_R^{-1}(K,\, M) = 0$, i.e., it should be 0.)

**Theorem.** *Let $R$ be a homomorphic image of a regular ring, and let $M$ be a finitely generated $R$-module. Let $t \geq 1$ be a fixed integer. Then the set*

$$\{P \in \mathrm{Spec}\,(R) : M_P \text{ is Cohen-Macaulay of type } \leq t\}$$

*is Zariski open in* $\mathrm{Spec}\,(R)$.

*Proof.* Let $R = S/J$, where $S$ is regular. Then $\mathrm{Spec}\,(R)$ is homeomorphic with the closed set $V(J) \subseteq \mathrm{Spec}\,(S)$: if we identify $\mathrm{Spec}\,(R)$ with $V(J)$, the locus we want in $\mathrm{Spec}\,(R)$ is the locus for $\mathrm{Spec}\,(S)$ intersected with $V(J)$. Thus, it suffices to consider the problem for $S$ instead, and we may assume without loss of generality that $R$ is regular.

Let $P$ be a prime of $R$ such that $M_P$ is Cohen-Macaulay of type at most $t$. If $M_P$ is 0, this will be true on a Zariski neighborhood of $P$, and we assume $M_P \neq 0$. By the preceding result, we may localize at one element of $R-P$ so that $M$ will be Cohen-Macaulay with annihilator of pure height $h$. Then $\mathrm{Ext}_R^h(M,\, R)_P \cong \mathrm{Ext}_{R_P}^h(M_P,\, R_P) = M_P^*$ can be generated by $t$ or fewer elements, and by clearing denominators we may assume that these elements have the form $u_1/1, \, \ldots\, , \, u_t/1$ where $u_i \in \mathrm{Ext}_R^h(M,\, R)$ for all $i$. (If fewer than $t$ generators are needed we may take some of the $u_i$ to be 0.) Let $N$ be the $R$-span of the $u_i$. Then $\bigl(\mathrm{Ext}_R^h(M,\, R)/N\bigr)_P = 0$ and so we can localize at one element of $R - P$ that kills $\mathrm{Ext}_R^h(M,\, R)/N$. After this localization, we have that $\mathrm{Ext}_R^h(M,\, R) = N$ is generated by at most $t$ elements, and so for all $Q$,

$$\mathrm{Ext}_R^h M,\, R)_Q \cong \mathrm{Ext}_{R_Q}^h(M_Q,\, R_Q) = M_Q^*$$

has at most $t$ generators. But this implies that the type of $M_Q$ is at most $t$, as required. $\square$

**Corollary.** *Let $R$ be a homomorphic image of a regular ring. Then*

$$\{P \in \mathrm{Spec}\,(R) : R_P \text{ is Gorenstein}\}$$

*is Zariski open in* $\mathrm{Spec}\,(R)$.

*Proof.* We may apply the preceding result with $M = R$. The fact that the type of $R_P$ is at most one implies that it is exactly one. $\square$

We have already proved for a local flat homomorphism $(R,\, m,\, K) \to (S,\, n,\, L)$ of local rings that $S$ is Cohen-Macaulay (respectively, Gorenstein) if and only if both $R$ and $S/mS$ are Gorenstein. We next want to give a global version of this result that also describes the behavior of the loci where these properties fail. We treat the Cohen-Macaulay and Gorenstein cases simultaneously by axiomatizing the properties we need.

Recall that if $R \to S$ is a ring homorphism, its fiber over $P \in \operatorname{Spec}(R)$ is $\kappa_P \otimes_R S$, where $\kappa_P = R_P/PR_P \cong \operatorname{frac}(R/P)$, the fraction field of $R/P$. Thus, the fiber may also be described as $(R-P)^{-1}S/PS$. The map $S \to (R-P)^{-1}S/PS$ induces an injection $\operatorname{Spec}\big((R-P)^{-1}S/PS\big) \hookrightarrow \operatorname{Spec}(S)$ whose image is the set of prime ideals of $S$ lying over $P$ in $R$. Thus, the primes in the spectrum of the fiber are in bijective correspondence with the prime ideals of $S$ that contract to $P$.

**Theorem.** *Let $\mathcal{P}$ denote a property of Noetherian rings such that:*
(1) *If a local ring $R$ has $\mathcal{P}$, so does its localization at any prime.*
(2) *$R$ has $\mathcal{P}$ if and only if its localization at every maximal ideal has $\mathcal{P}$ (it then follows that all of its localizations have $\mathcal{P}$).*
(3) *If $(R, m, K) \to (S, n, L)$ is a local map of local rings, then $S$ has $\mathcal{P}$ if and only if $R$ has $\mathcal{P}$.*

*Then the following statements hold:*
(a) *If $S$ is faithfully flat over $R$, then $S$ has $\mathcal{P}$ if and only if $R$ has $\mathcal{P}$ and every fiber of $R \to S$ has $\mathcal{P}$.*
(b) *If $R \to S$ is flat, all of the fibers have $\mathcal{P}$, and $I$ is an ideal of $S$ such that $V(I)$ is the set of primes of $R$ that do not have property $\mathcal{P}$, then $V(IS)$ is the set of primes of $S$ that do not have property $\mathcal{P}$.*

*In particular, these results hold when $\mathcal{P}$ is the property of being Cohen-Macaulay and when $\mathcal{P}$ is the property of being Gorenstein.*

*Proof.* We first consider part (a). Assume that $S$ has $\mathcal{P}$. For every prime $P$ of $R$ there is a prime $Q$ of $S$ lying over $P$. Since $S_Q$ has $\mathcal{P}$, so does $R_P$. Therefore, $S$ has $\mathcal{P}$ implies that $R$ has $\mathcal{P}$. Each prime of the fiber $(R-P)^{-1}(S/PS)$ corresponds to a prime $Q$ of $S$ lying over $P$, and it suffices to show that every ring $\big((R-P)^{-1}(S/PS)\big)_Q$ has $\mathcal{P}$. But this ring $\cong S_Q/PS_Q$, which has $\mathcal{P}$ because $S$ does.

Now assume that $R$ and all fibers have $\mathcal{P}$. Let $Q$ be a prime of $S$ lying over $P$ in $R$. It suffices to show that $S_Q$ has $P$. This is true because $R_P$ and $S_Q/PS_Q$ both have $\mathcal{P}$: the latter is a localization of $(R-P)^{-1}S/PS$.

To prove (b), let $Q$ be a prime ideal of $S$ and let $P$ be its contraction to $R$. Note that $Q \in V(IS) \Leftrightarrow P \in V(I)$. If $S_Q$ has $\mathcal{P}$, so does $R_P$, and so $P \notin V(I)$ and $Q \notin V(IS)$. If $Q \in V(IS)$ then $P \in V(I)$, so that $R_P$ does not have $\mathcal{P}$ and $S_Q$ does not have $\mathcal{P}$. $\square$

Note that when $R$ is a Hodge algebra over $K$ on $H$ governed by $\Sigma$, arbitrary base change on $K$ produces a new Hodge algebra with the same data. More precisely, if $K \to K'$ is any ring homomorphism, $R' = K' \otimes_K R$, $H'$ is the image of $H$ in $R'$ under the map sending $h \mapsto 1 \otimes h$, and $\Sigma'$ is the semigroup corresponding to $\Sigma$ under the obvious isomorphism $\mathbb{N}^H \to \mathbb{N}^{H'}$, then $R'$ is a Hodge algebra over $K'$ on $H'$ governed by $\Sigma'$. The free basis of standard monomials for $R$ evidently maps bijectively to a free basis for $R'$ over $K'$, and the straightening relations for $R$ map to the required straightening relations for $R'$. In particular, each fiber $\kappa_P \otimes_K R$ is a Hodge algebra over a field.

**Corollary.** *A Hodge algebra over a Noetherian ring $K$ is Cohen-Macaulay (respectively, Gorrenstein) if and only if $K$ is Cohen-Macaulay (respectively, Gorenstein) and each fiber* *is Cohen-Macaulay (respectively, Gorenstein). The same holds for any property of rings $\mathcal{P}$ satisfying the three conditions in the Theorem above.*

*The condition that each fiber is Cohen-Macaulay (respectively, Gorenstein) is equivalent to the condition that for every field $\kappa$ to which $K$ maps, $\kappa \otimes_K R$ is Cohen-Macaulay (respectively, Gorenstein).*

*Proof.* The Hodge algebra is a free over $K$ on a basis containing 1, and is therefore faithfully flat over $K$. The result is immediate from part (a) of the Theorem just above. The final statement follows from the fact that if $P$ is the kernel of $K \to \kappa$ the map to $\kappa$ factors through the fiber $K_P/PK_P$. The final statement now follows from the Lemma just following. $\square$

**Lemma.** *Let $B$ be a finitely generated $\kappa$-algebra. Then $B$ is Cohen-Macaulay (respectively, Gorenstein) if and only if $B' = \kappa' \otimes_\kappa B$ has the specified property for every field extension $\kappa'$ of $\kappa$.*

*Proof.* $B'$ is faithfully flat over $B$ and so the "if" part follows. Now assume that $B$ has the specified property. The result will follow if each fiber is Gorenstein (the fibers are then Cohen-Macaulay as well). Each fiber has the form $\kappa' \otimes_\kappa L$ where $L$ has the form $B_P/PB_P$ and so is a field finitely generated over $\kappa'$. We proceed by induction on the number of generators of the field $L$ over $\kappa$. If $\kappa \subseteq L_0 \subseteq L$, we have that

$$\kappa' \otimes_\kappa L \cong (\kappa' \otimes_\kappa L_0) \otimes_{L_0} L.$$

Therefore, it suffices to show that if $C$ is a Gorenstein algebra containing a field $L_0$ and $L$ is s field generated over $L_0$ by one element, then $D = C \otimes_{L_0} L$ is Gorenstein. There are two cases. If $L = L_0(x)$ where $x$ is transcendental over $L_0$, then $C \otimes_{L_0} L$ is a localization of $C[x]$, and this is Gorenstein, since it is flat over $C$ with Gorenstein fibers. If $L$ is generated by one element $\theta$ over $L_0$ and is algebraic, let $f$ be the minimal monic polynomial of $\theta$ over $L_0$. Then $D \cong C \otimes_{L_0} L_0[x]/(f) \cong C[x]/(f)$. But $C[x]$ is Gorenstein, and the monic polynomial $f$ is a nonzerodivisor. Thus, the quotient is also Gorenstein. $\square$

The following theorem gives that the defining radical ideal of the closed set of primes where a graded ring is not Cohen-Macaulay or not Gorenstein is homogeneous. We need a preliminary fact.

**Lemma.** *If $T$ is flat over a reduced ring $R$ and the fibers are reduced then $T$ is reduced.*

*Proof.* If $T$ has a nilpotent element other than 0, we may localize at a minimal prime $Q$ of its annihilator, and if $Q$ lies over $P$ we may study $R_P \to T_Q$ instead. Then $T_Q$ has depth 0, and so $R_P$ has depth 0. Since this ring is reduced and local, it must be a field. But then $T_Q$ is the fiber over $P$ (a localization of the original fiber over $P$) and is reduced. $\square$

**Theorem.** *Let $S \subseteq \mathbb{N}^h$ be a semigroup and let $R$ be a Noetherian ring graded by $S$. Suppose that the set of primes such that $R_P$ is not Cohen-Macaulay (respectively, not Gorenstein) is closed. Then the radical ideal $I$ defining this locus is homogeneous in the $S$-grading.*

*Proof.* There is no loss of generality in assuming that $S = \mathbb{N}^h$: we can enlarge $S$, and define the new graded pieces to be 0. For each $i$, $1 \le i \le h$, we can put a $\mathbb{Z}$-grading on

$R$ as follows: let $R_t^{(i)}$ be the sum of all components $R_s$ of $R$ such that the $i$ th coordinate of $s$ is $t$. It will suffice to show the result when $h = 1$, for if $u$ is an element of $I$ and $(s_1, \ldots, s_h) \in \mathbb{N}^h$, we define recursively a sequence of elements $v_0 = u, v_1, \ldots, v_h$ by letting $v_{j+1}$ be the $s_j$ component of $v_j$ with respect to the $j$ th $\mathbb{Z}$-grading. By induction, each of the $v_j$ is in $I$, and $v_h$ is $u_s$. Thus, every $S$-homogeneous component of every element of $I$ is in $I$.

It remains to handle the case where $S = \mathbb{Z}$, i.e., $h = 1$. Let $u \in I$ and suppose $u = u_{a+1} + \cdots + u_{a+n}$ is the decomposition of $u$ as a sum of forms: here $u_i$ has degree $i$.

If $\alpha$ is a unit of $R_0$, we can define an endomorphism $\theta_\alpha : R \to R$ by letting it act on $R_s$ by mutliplication by $\alpha^s$, where $s \in \mathbb{Z}$. This clearly gives an $R_0$-linear map $R \to R$, and one can see easily that multiplication is preserved because degrees add when one multiplies forms. Moreover, if $\beta = \alpha^{-1}$, then $\theta_\alpha$ and $\theta_\beta$ are inverses, and so each $\theta_\alpha$ is an automorphism of $R$. Every automorphism of $R$ must map $I$ to itself.

If $R_0$ contains $n$ distinct units $\alpha_j$ such that the differences $\alpha_i - \alpha_j$ for $i \neq j$ are also units, then we obtain for each $j$ that

$$\sum_{i=1}^{n} \alpha_j^{n+i} u_j = r_j$$

where $r_j \in I$. The result now follows because the $n \times n$ matrix $A = (\alpha_j^{n+i})$ is invertible. To see this, We may factor the unit $\alpha_j^{n+1}$ from the $j$ th column. It therefore suffices to see that the Van der Monde matrix $(\alpha_j^{i-1})$ is invertible. This follows because the determinant of this matrix is $D = \prod_{i<j}(\alpha_j - \alpha_i)$. It suffices to prove this when the $\alpha_j$ are indeterminates over the integers $\mathbb{Z}$. In this case note that if $\alpha_i = \alpha_j$ two columns are equal and the determinant vanishes. Thus, the determinant is divisible by every $\alpha_j - \alpha_i$. Since these are relatively prime in pairs and the polynomial ring is a UFD, the determinant is a multiple of $D$. The degree of both $D$ and the determinant is $\binom{n}{2}$. Therefore, the multiplier is a constant. The product of the terms on the main diagonal is the same as the product of the first terms from factors $\alpha_j - \alpha_i$ for $i < j$, and this term does not occur elsewhere in either expansion. Therefore the multiplier is 1.

It remains to consider the case where $R_0$ does not have sufficiently many units as described. Again, fix $u$ as described above in $I$, with at most $n$ consecutive possibly nonzero homogeneous components. Adjoin $n$ indeterminates $z_j$ to $R_0$ and localize at the element $g$ which is the product of the $z_j$ and the $z_j - z_i$ for $j \neq i$. The resulting ring $R_0'$ is faithfully flat over $R_0$. Then $T = R_0' \otimes_{R_0} R \cong R[z_1, \ldots, z_d]_g$, and each fiber over $R$ is a localized polynomial ring. It follows that $IT$ is a defining ideal of the locus of primes of $T$ lacking the specified property, and $IT$ is still radical (by the preceding Lemma applied to $R/I$ and $T/IT$) and therefore mapped to itself by every automorphism of $T$. The argument given above shows that every homogeneous component of $u$ is in $IT \cap R = I$, by the faithful flatness of $T$ over $R$. $\square$