# Math 711: Lecture of October 10, 2005

We can now prove the following result: it was asserted without proof in the Theorem at the bottom of the second page of the Lecture Notes of September 19.

**Proposition.** *An $\mathbb{N}$-graded algebra $R$ finitely generated over $R_0 = K$, where $K$ is a field, is Cohen-Macaulay (respectively, Gorenstein) if and only if that is true for the local ring at the homogeneous maximal ideal $m$.*

*Proof.* Obviously, if $R$ is Cohen-Macaulay then $R_m$ is. If $R$ is not Cohen-Macaulay then the radical defining ideal of the non-Cohen-Macaulay locus is homogeneous. If it is proper, it is contained in $m$, and so $R_m$ is not Cohen-Macaulay. $\square$

**Corollary.** *Let $R$ be $\mathbb{N}$-graded and finitely generated over $R_0 = K$, a field. Let $m$ be the homogeneous maximal ideal of $R$. If $I \subseteq m$ and $\mathrm{gr}_I R$ is Cohen-Macaulay (respectively, Gorenstein), so is $R$.*

*Proof.* We know that $R_Q$ has the specified property for every prime ideal $Q \subseteq I$, by the the Theorem on the third page of the Lecture Notes of September 28, and so $R_m$ has the specified property, which implies that $R$ does. $\square$

We then have:

**Theorem.** *Let $R$ be a Hodge algebra over $K$ on $H$ governed by $\Sigma$. A sufficient condition for $R$ to be Cohen-Macaulay (respectively, Gorenstein) is that $K$ be Cohen-Macaulay (respectively, Gorenstein) and that for every map $K \to \kappa$, where $\kappa$ is a field, the corresponding discrete Hodge algebra over $\kappa$ be Cohen-Macaulay.*

*Proof.* We already know that it suffices if $K$ and each $\kappa \otimes R$ has the specified property. Therefore, we need only prove that when $K = \kappa$ is a field, if the discrete Hodge algebra has the property, then so does $R$. But, starting with $R$, there is a sequence of $\mathbb{N}$-graded rings with degree 0 component equal to $K$, each finitely generated over $K$, each the associated graded ring of its predecessor in the sequence with respect to an ideal generated by a form of positive degree, and such that the last ring in the sequence is the discrete Hodge algebra. It follows by iterated use of the preceding Corollary that all of these rings have the specified property. $\square$

Note that when $R$ is an ASL, the discrete Hodge algebras over fields are face rings over a field.

We want to be able to determine the Krull dimension of a Hodge algebra. If $H$ is a finite poset and $\Sigma \subseteq \mathbb{N}^H$ is a semigroup ideal, let $\Delta_\Sigma$ denote the simplicial complex, with vertices in the set $H$, whose simplices are the subsets of $H$ that are not the support of any element of $\Sigma$. If $K$ is a field, and one considers the discrete Hodge algebra on $H$ governed by $\Sigma$, say $K[H]/J$, where $J$ is generated by monomials with exponent in $\Sigma$, then it is easy to see that $(K[H]/J)_{\mathrm{red}} \cong K[\Delta_\Sigma]$.

**Proposition.** *Let $R$ be a Hodge algebra over $K$ on $H$ governed by $\Sigma$. Then* $\dim(R) = \dim(K) + \dim(\Delta_\Sigma) + 1$.

*Proof.* Any prime $Q$ of $R$ lies over a prime $P$ of $K$, and the dimension of $R_Q$ is the sum of the dimensions of $K_P$ and of the fiber over $P$ localized at $Q$. The result now follows from the fact that all fibers have the same Krull dimension: the fiber over $P$ is the discrete Hodge algebra over $\kappa_P$ on $H$ governed by $\Sigma$, and the corresponding reduced ring has the same dimension as $\kappa_P[\Delta_\Sigma]$, which is $\dim(\Delta_\Sigma) + 1$. □

We next want to discuss the notion of an F-injective Cohen-Macaulay ring in characteristic $p > 0$. We shall not define F-injectivity if the ring is not Cohen-Macaulay.

**Proposition.** *Let $(R, m, K)$ be a local ring of positive prime characteristic $p$ or else let $R$ be an $\mathbb{N}$-graded finitely generated algebra over a field $K$ of characteristic $p > 0$ with $R_0 = K$. Assume that $R$ is Cohen-Macaulay of Krull dimension $d$. The following conditions are equivalent:*

(1) *For every sequence of elements $x_1, \ldots, x_k$ that is part of a system of parameters (a homogeneous system of parameters in the graded case), if $u^p \in (x_1^p, \ldots, x_k^p)$ then $u \in (x_1, \ldots, x_k)R$.*
(2) *For every sequence of elements $x_1, \ldots, x_d$ that is a system of parameters (a homogeneous system of parameters in the graded case), if $u^p \in (x_1^p, \ldots, x_d^p)$ then $u \in (x_1, \ldots, x_d)R$.*
(3) *For every sequence of elements $x_1, \ldots, x_k$ that is part of a system of parameters (a homogeneous system of parameters in the graded case), if $q = p^e$ is a power of $p$ and if $u^q \in (x_1^q, \ldots, x_k^q)$, then $u \in (x_1, \ldots, x_k)R$.*
(4) *For every sequence of elements $x_1, \ldots, x_d$ that is a system of parameters (a homogeneous system of parameters in the graded case), if $q = p^e$ is a power of $p$ and if $u^q \in (x_1^q, \ldots, x_d^q)$, then $u \in (x_1, \ldots, x_d)R$.*

*If these equivalent conditions hold, we say that $R$ is F-injective.*

*Proof.* (3) obviously implies (4) and (1), both of which imply (2). Therefore it suffices to show that (2) implies (1) and that (1) implies (4). Given $x_1, \ldots, x_k$ extend it to a full (and, in the graded case, homogeneous) system of parameters $x_1, \ldots, x_d$. Then $(x_1, \ldots, x_k, x_{k+1}^N, \ldots, x_d^N)$ is an ideal generated by a full system of parameters, and since

$$u^p \in (x_1^p, \ldots, x_k^p, x_{k+1}^{pN}, \ldots, x_d^{pN})R,$$

we have that

$$u \in (x_1, \ldots, x_k, x_{k+1}^N, \ldots, x_d^N)R$$

for all $N$. Since every ideal is $m$-adically closed in a local ring (in the graded case, this is true for homogeneous ideals by an easy degree argument), we have that $u \in (x_1, \ldots, x_k)R$. The fact that (1) implies (4) is easy by induction on $e$. The case $e = 1$ is given. For the inductive step, we have that

$$(u^{p^{e-1}})^p \in \big((x_1^{p^{e-1}})^p, \ldots, (x_k^{p^{e-1}})^p\big)R,$$

whence, by the case $e = 1$, we have that

$$u^{p^{e-1}} \in \bigl(x_1^{p^{e-1}}, \dots, x_k^{p^{e-1}}\bigr)R,$$

and the result is immediate from the induction hypothesis. $\square$

We shall soon prove that it suffices to check what happens with just one system of parameters.