Math 711: Lecture of October 12, 2005

Discussion. Let (R, m, K) be a Cohen-Macaulay ring of Krull dimension d. We want to define an R-module $H(R)$ associated canonically with R. (For those familiar with local cohomology, it will turn out that $H(R) = H^d_m(R)$. We shall attempt to avoid making use of any substantial knowledge of local cohomology theory.)

The key point is that if $x_1, \ldots, x_d$ and $y_1, \ldots, y_d$ are two systems of parameters for R, if $(x_1, \ldots, x_d)R \supseteq (y_1, \ldots, y_d)R$ then there is a canonical map $R/(x_1, \ldots, x_d)R \to R/(y_1, \ldots, y_d)$. This should not be confused with the obvious surjection $R/(y_1, \ldots, y_d) \twoheadrightarrow R/(x_1, \ldots, x_d)$ in the other direction: in fact, we shall eventually show that the maps $R/(x_1, \ldots, x_d)R \to R/(y_1, \ldots, y_d)R$ are injective.

The map is constructed by choosing a $d \times d$ matrix $A = (r_{ij})$ such that $Y = XA$, where Y is the $1 \times d$ row matrix whose entries are $y_1, \ldots, y_d$, and X is the $1 \times d$ row matrix whose entries are $x_1, \ldots, x_d$. The existence of A is entirely equivalent to the fact that each x_j is an R-linear combination of the elements $y_1, \ldots, y_d$. The map $R/(x_1, \ldots, x_d) \to R/(y_1, \ldots, y_d)$ is then induced by multiplication by $\delta = \det(A)$ acting on the numerators.

Let $\mathrm{adj}(A)$ denote the transpose of the cofactor matrix of A, the *classical adjoint* of A. A standard identity yields that $A\big(\mathrm{adj}(A)\big) = \delta I_d$, where I_d is $d \times d$ identity matrix. Since $Y = XA$, multiplying both sides on the right by $\mathrm{adj}(A)$ yields that $Y\mathrm{adj}(A) = \delta X$, which shows that multiplication by δ takes $(x_1, \ldots, x_d)R$ into $(y_1, \ldots, y_d)R$. This shows that multiplication by δ does induce a map $R/(x_1, \ldots, x_d)R \to R/(y_1, \ldots, y_d)R$.

We next want to show that this map is independent of the choice of the matrix A.

We first recall some facts about the Koszul complex: the point of view we shall take, which is the exterior algebra point of view, is discussed in the Lecture Notes of March 1 from Math 615, Fall 2004.

Consider the Koszul complex $\mathcal{K}_\bullet(x_1, \ldots, x_d;\, R)$ of $x_1, \ldots, x_d$ on R. If $G = \bigoplus_{j=1}^{d} Ru_j = \mathcal{K}_1(x_1, \ldots, x_d;\, R)$ is the free module on the free basis $u_1, \ldots, u_d$ and the differential takes $u_j \mapsto x_j$, $1 \le j \le d$, then the differential on the whole complex is the uniqe extension of this map to an exterior algebra derivation on $\bigwedge^\bullet(G)$. The matrix A induces a map of Koszul complexes:

$$\begin{array}{ccccccccccccc}
0 & \longrightarrow & \bigwedge^d G & \longrightarrow & \bigwedge^{d-1} G & \longrightarrow & \cdots & \longrightarrow & R^d & \xrightarrow{X} & R & \longrightarrow & 0 \\
 & & \bigwedge^d A \Big\uparrow & & \bigwedge^{d-1} A \Big\uparrow & & & & \Big\uparrow A & & \Big\uparrow \mathrm{id} & & \\
0 & \longrightarrow & \bigwedge^d G & \longrightarrow & \bigwedge^{d-1} G & \longrightarrow & \cdots & \longrightarrow & R^d & \xrightarrow{Y} & R & \longrightarrow & 0
\end{array}$$

while if we replace A by another choice A' such that $Y = XA'$ with $\det(A') = \delta'$, we get another such map of complexes. Since the top row is acyclic and the bottom row consists of projective modules, the two maps of complexes are homotopic: the needed facts about homotopy may be found in the Lecture Notes of February 2 and February 4 from Math

615, Fall 2004. But all we need to know is that this implies that the difference $\bigwedge^d A - \bigwedge^d A'$ of the two maps of complexes in degree d has the form $h \circ d$ where

$$h : \mathcal{K}_{d-1}(y_1, \ldots, y_d;\, R) \to \mathcal{K}_d(x_1, \ldots, x_d;\, R)$$

and

$$d : \mathcal{K}_d(y_1, \ldots, y_d;\, R) \to \mathcal{K}_{d-1}(y_1, \ldots, y_d; R)$$

is the next to last nonzero map in the Koszul complex $\mathcal{K}_\bullet(y_1, \ldots, y_; R)$. We may identify the leftmost two nonzero terms in the two Koszul complexes with R and R^d respectively. When we do so, the vertical maps $\bigwedge^d A$ and $\bigwedge^d A'$ are identified with multiplication by $\det(A) = \delta$ and $\det(A') = \delta'$, respectively, and the matrix of the map $d : R \to R^d$ has entries which are, up to sign, the y_j. The existence of the homotopy shows therefore shows that $\delta - \delta' \in (y_1, \ldots, y_d)R$. It follows that the maps $R/(x_1, \ldots, x_d)R \to R/(y_1, \ldots, y_d)R$ induced by multiplication by δ and multiplication by δ' are the same.

Let (R, m, K) be a Cohen-Macaulay local ring. Whenever we have an containment $(x_1, \ldots, x_d)R \supseteq (y_1, \ldots, y_d)R$ we have a canonical map

$$R/(x_1, \ldots, x_d)R \to R/(y_1, \ldots, y_d)R.$$

These maps depend, however, on knowing the choices of parameters, not just on the ideals. For example, when the systems of parameters are x_1, x_2 and x_2, x_1 the map $R/(x_1, x_2)R \to R/(x_2, x_1)R$ is multiplication by -1, not the identity map.

If $z_1, \ldots, z_d$ is a third system of parameters such that $(y_1, \ldots, y_d) \supseteq (z_1, \ldots, z_d)R$ we have maps $R/(x_1, \ldots, x_d)R \to R/(y_1, \ldots, y_d)R$ and $R/(y_1, \ldots, y_d)R \to R/(z_1, \ldots, z_d)R$: their composition is the map $R/(x_1, \ldots, x_d)R \to R/(z_1, \ldots, z_d)R$ determined by the systems of parameters $z_1, \ldots, z_d$ and $x_1, \ldots, x_d$ and the containment $(x_1, \ldots, x_d)R \supseteq (z_1, \ldots, z_d)R$. The point is that if $X = YA$ and $Y = ZB$, then $X = (ZB)A = Z(BA)$, and $\det(AB) = \det(A)\det(B)$.

We next prove that the map $R/(x_1, \ldots, x_d)R \to R/(y_1, \ldots, y_d)R$ is injective. To see this, choose $N \gg 0$ such that $(y_1, \ldots, y_d)R \supseteq (x_1^N, \ldots, x_d^N)R$. To show that $R/(x_1, \ldots, x_d)R \to R/(y_1, \ldots, y_d)R$ is injective, it suffices to show that its composition with $R/(y_1, \ldots, y_d)R \to R/(x_1^N, \ldots, x_d^N)R$ is injective, and this is the map

$$R/(x_1, \ldots, x_d)R \to R/(x_1^N, \ldots, x_d^N)R.$$

To see that this map is injective, note that we may choose for the matrix A the diagonal matrix whose j th diagonal entry is x_j^{N-1}. The map $R/(x_1, \ldots, x_n) \to R/(x_1^N, \ldots, x_d^N)R$ is induced by multiplication by the determinant of A, which is $x_1^{N-1} \cdots x_d^{N-1}$. The injectivity of the map then reduces to the assertion that $(x_1^N, \ldots, x_d^N)R :_R x_1^{N-1} \cdots x_d^{N-1} = (x_1, \ldots, x_d)R$. This follows from the fact that $x_1, \ldots, x_d$ is a regular sequence on R. The following Lemma establishes this:

Lemma. *Let $x_1, \ldots, x_d$ be a (possibly improper) regular sequence on an R-module M.*
(a) *If $u_1, \ldots, u_d \in M$ are such that $\sum_{i=1}^d x_j u_i = 0$ then for every j,*

$$u_j \in (x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_d)M.$$

(b) *Let* $a_1, \ldots, a_d$ *be positive integers and* $b_1, \ldots, b_d$ *be nonnegative integers. Then* $(x_1^{a_1+b_1}, \ldots, x_d^{a_d+b_d})M :_M \prod_{j=1}^d x_j^{b_j} = (x_1^{a_1}, \ldots, x_d^{a_d})M$.

Proof. For part (a), if $d = 1$ or $j = d$, the result is immediate from the definition of a regular sequence. Use induction on d and assume $j < d$. Then we have $u_d = \sum_{i=1}^{d-1} x_i v_i$, and we can substitute to obtain that $\sum_{i=1}^{d-1} x_i(u_i + x_d v_i) = 0$. The induction hypothesis yields that $u_j + x_d v_j$ is in the ideal generated by the other x_i times M, and the result follows.

For part (b), note that $M : (IJ) = (M :_M I) :_M J$, since $IJu \subseteq M$ iff $Ju \subseteq M :_M I$ iff $u \in (M :_M I) :_M J$. This extends in the obvious way to any finite product of ideals. We therefore only need to prove (b) when at most one of the b_i, say b_j, is nonzero. The case where $b_j = 0$ is obvious and we assume that $b_j > 0$. But if

$$x_j^{b_j} u = \sum_{i=1}^{d} x_i^{a_i+b_i} u_i$$

(here, only b_j is nonzero) we can move the term $x_j^{b_j} u$ on the left to the other side, altering the j th term in the summation to become $x_j^{b_j}(x_j^{a_j} u_j - u)$, while the other terms are unaffected. We may now apply (a) to get that $x_j^{a_j} u_j - u$ is in the ideal generated by the $x_i^{a_i}$ for $i \neq j$ times M, which readily yields $u \in (x_1^{a_1}, \ldots, x_d^{a_d})M$, as required. □

Under the injections $R/(x_1, \ldots, x_d) \hookrightarrow R/(y_1, \ldots, y_d)$ the socle must map into the socle. Since this is an injective map of finite-dimensional K-vector spaces of the same dimension, the map induces an isomorphism of one socle with the other.

Let $\mathcal{S}$ be the set of (ordered) systems of parameters for R. We may now use the maps that we have constructed above to build a direct limit:

$$\varinjlim_{x_1, \ldots, x_d \in \mathcal{S}} \frac{R}{(x_1, \ldots, x_d)R}.$$

We denote the limit $H(R)$. Note that every $R/(x_1, \ldots, x_d)R$, where $x_1, \ldots, x_d$ is a system of parameters, embeds in $H(R)$. Also note that if $x_1, \ldots, x_d$ and $y_1, \ldots, y_d$ are two systems of parameters, then $R/(y_1, \ldots, y_)R$ embeds in $R/(x_1^t, \ldots, x_d^t)R$ for all $t \gg 0$. Therefore, we may fix a system of parameters $x_1, \ldots, x_d$ and $H(R) \cong \varinjlim_t R/(x_1^t, \ldots, x_d^t)R$. The maps between consecutive terms in this latter direct limit system are induced by multiplication by $x_1 \cdots x_d$.

The maps we have constructed may be viewed in another way. If $x_1, \ldots, x_d$ is a system of parameters in a regular ring, then the Koszul complex on $x_1, \ldots, x_d$ may be used to identify $\mathrm{Ext}_R^d(R/(x_1, \ldots, x_d)R, R)$ with $R/(x_1, \ldots, x_d)R$. When $(x_1, \ldots, x_d)R \subseteq (y_1, \ldots, y_d)R$ the surjection $R/(y_1, \ldots, y_d)R \twoheadrightarrow R/(x_1, \ldots, x_d)R$ induces a map

$$\mathrm{Ext}_R^d(R/(x_1, \ldots, x_d)R, R) \to \mathrm{Ext}_R^d(R/(x_1, \ldots, x_d)R, R).$$

After identifying the first module with $R/(x_1, \dots, x_d)R$ and the second module with $R/(y_1, \dots, y_d)R$, this is the map we constructed.

We next want to use $H(R)$ to study F-injective Cohen-Macaulay local rings. Let R be a ring of positive prime characteristic p and let M be an R-module. By an *action of Frobenius* F *on* M we mean a $\mathbb{Z}$-linear map $F : M \to M$ such that for all $r \in R$ and $u \in M$, $F(rm) = r^p F(m)$.

When R is Cohen-Macaulay local there is a standard action of F on $H(R)$. If $r \in R$ and $x_1, \dots, x_d$ is a system of parameters for R, let $(r; x_1, \dots, x_d)$ denote the image of r in $R/(x_1, \dots, x_d)$ and, hence, in $H(R)$. Every element of $H(R)$ has this form. We let F act by sending $(r; x_1, \dots, x_d) \mapsto (r^p; x_1^p, \dots, x_d^p)$. It is easy to check that F is well-defined and gives an action of Frobenius on $H(R)$. We can now prove:

Theorem. *The following conditions on a Cohen-Macaulay local ring R of prime characteristic $p > 0$ are equivalent.*
(a) *R is F-injective.*
(b) *$F : H(M) \to H(M)$ is injective.*
(c) *There exists a system of parameters $x_1, \dots, x_d$ for R such that if $u \in R$ and $u^p \in (x_1^p, \dots, x_d^p)R$ then $u \in (x_1, \dots, x_d)R$.*
(d) *There exists a system of parameters $x_1, \dots, x_d$ for R such that if $u \in R$ represents an element of the socle of $R/(x_1, \dots, x_d)$ and $u^p \in (x_1^p, \dots, x_d^p)R$ then $u \in (x_1, \dots, x_d)R$.*

Proof. The equivalence of (a) and (b) is clear, since the action of f on $(r; x_1, \dots, x_d)$ maps it to $(r^p; x_1^p, \dots, x_d^p)$ and this will be zero iff $r^p \in (x_1^p, \dots, x_d^p)R$. It is clear that (a) implies (c) and that (c) implies (d). But (c) is equivalent to (d), for if one has $u \in R - (x_1, \dots, x_d)R$ such that $u^p \in (x_1^p, \dots, x_d^p)R$, one may replace u be a multiple that represents an element of the socle in $R/(x_1, \dots, x_d)$. It therefore suffices to prove that (c) implies (b).

Here, we make use of the fact that $H(R)$ is the direct limit of the submodules $R/(x_1^t, \dots, x_d^t)$. Hence, we may assume that if some element is killed by F, it has the form $(r; x_1^t, \dots, x_d^t)$. Moreover, we may assume that r represents an element of the socle mod $(x_1^t, \dots, x_d^t)R$ (replacing it by a multiple if necessary), and therefore we may assume that it has the form $x_1^{t-1} \cdots x_d^{t-1} u$, where u represents an element in the socle of $R/(x_1, \dots, x_d)R$. We then find that $(x_1^{t-1} \cdots x_d^{t-1} u)^p \in (x_1^{pt}, \dots, x_d^{pt})R$ and so $u^p \in (x_1^p, \dots, x_d^p)R :_R x_1^{pt-p} \cdots x_d^{pt-p} = (x_1^p, \dots, x_d^p)R$ by part (b) of the Lemma. But then $u \in (x_1, \dots, x_d)R$, and $(u; x_1, \dots, x_d) = 0$. $\square$

Parallel to this we have a graded result. Let R be a finitely generated $\mathbb{N}$-graded algebra over $R_0 = K$, a field. Let d be the Krull dimension of R. Let $\mathcal{S}_h$ denote the set of homogeneous systems of parameters of R. We can define

$$H(R) = \varinjlim_{x_1, \dots, x_d \in \mathcal{S}_h} R/(x_1, \dots, x_d)R$$

exactly as in the local case, although here we have limited the systems of parameters to be homogeneous. It is easy to check that $H(R)$ has a $\mathbb{Z}$-grading such that the degree of $(r; x_1, \dots, x_d)$, where r and $x_1, \dots, x_d$ are homogeneous, is $\deg(r) - \sum_{j=1}^d \deg(x_j)$. Moreover, if the field has characteristic $p > 0$ there is an action of Frobenius on $H(R)$ that multiplies degrees by p, defined exactly as in the local case: $F(r; x_1, \dots, x_d) = (r^p; x_1^p, \dots, x_d^p)$.

Theorem. *Let R be a finitely generated $\mathbb{N}$-graded algebra over $R_0 = K$, a field of characteristic $p > 0$. Let m be the homogeneous maximal ideal. Suppose that R is Cohen-Macaulay. Then $H(R) \cong H(R_m)$. Moreover, the following conditions are equivalent:*

(a) *R is F-injective.*
(b) *F acts injectively on $H(R)$.*
(c) *R_m is F-injective.*
(d) *There exists a homogeneous system of parameters $x_1, \dots, x_d$ for R such that if $u \in R$ and $u^p \in (x_1^p, \dots, x_d^p)R$ then $u \in (x_1, \dots, x_d)R$.*
(e) *There exists a homogeneous system of parameters $x_1, \dots, x_d$ for R such if $u \in R$ represents an element of the annihilator of m in $R/(x_1, \dots, x_d)$ and $u^p \in (x_1^p, \dots, x_d^p)R$ then $u \in (x_1, \dots, x_d)R$.*

Proof. Fix a homogeneous system of parameters $x_1, \dots, x_d$. For every t, $R/(x_1^t, \dots, x_d^t)R$ has a unique maximal ideal, the image of m, and so $R/(x_1^t, \dots, x_d^t)R \cong R_m/(x_1^t, \dots, x_d^t)R_m$. Taking the direct limits over t of both sides gives that $H(R) \cong H(R_m)$.

It is immediate from the definition of F-injectivity for R that R is F-injective if and only if F acts injectively on $H(R)$, and we know the corresponding fact for R_m and $H(R_m)$. Since $H(R) \cong H(R_m)$ and the Frobenius actions are the same, the equivalence of (a), (b), and (c) follows. Conditions (d) and (e) imply the corresponding condition for R_m, since localization at m does not affect $R/(x_1, \dots, x_d)R$, and so the equivalence of (d) with (e) and the other conditions follows from the preceding Theorem. $\square$

Note that in the Gorenstein case the socle mod a system of parameters $x_1, \dots, x_d$ is isomorphic with K. Any element u of R which is nonzero in the socle is a unit times any other such element, and F-injectivity may be proved by simply checking that $u^p \notin (x_1^p, \dots, x_d^p)$ for this single choice of $x_1, \dots, x_d$ and u.

We have the following consequence of the fact that injectivity can be checked using a single system of parameters.

Theorem. *Let (R, m, K) be a local ring or let R be a finitely generated algebra over $R_0 = K$, a field, and assume that R is Cohen-Macaulay of prime characteristic $p > 0$. Let x be an element that is part of a system of parameters for R: in the graded case, assume that x is homogeneous. If R/xR is F-injective, then so is R.*

Proof. Extend x to a full system of parameters $x, x_2, \dots, x_d$ for R, homogeneous in the graded case. Let $u \in R$. We must show that if $u^p \in (x^p, x_2^p, \dots, x_d^p)$ then $u \in (x, x_2, \dots, x_d)R$.

We use an overline to indicate images in R/xR. Then $\overline{u}^p \in (\overline{x}_2^p, \dots, \overline{x}_d^p)$, and so $\overline{u} \in (\overline{x}_2, \dots, \overline{x}_d)$, from which $u \in (x, x_2, \dots, x_d)$ is immediate. $\square$