Math 711: Lecture of October 14, 2005

We next observe:

Proposition. *Let R be a Cohen-Macaulay local ring of characteristic p and suppose that R is F-injective.*
(a) *For every prime ideal P of R, R_P is Cohen-Macaulay and F-injective.*
(b) *If $A \to R$ is a flat local homomorphism, then A is Cohen-Macaulay and F-injective.*

Proof. For part (a), suppose that P has height k. We can choose $x_1, \ldots, x_k \in P$ that are part of a system of parameters for R. Their images will be a system of parameters for R_P. Now suppose that $u \in R$ is such that $u/1 \in (x_1^p, \ldots, x_k^p)R_P$ (we may assume that $u \in R$, since every element of R_P is a unit times an element of R). Then we can choose $w \in R - P$ such that $wu^p \in (x_1^p, \ldots, x_k^p)R$, and it follows that $(wu)^p \in (x_1^p, \ldots, x_k^p)R$ as well. But then $wu \in (x_1, \ldots, x_k)R$, and so $u \in (x_1, \ldots, x_d)R_P$. This proves part (a).

For part (b), note that we immediately know that A is Cohen-Macaulay. Let $x_1, \ldots, x_k$ be a system of parameters for A, and suppose that $u \in A$ is such that $u^p \in (x_1^p, \ldots, x_k^p)A$. Then the images of $x_1, \ldots, x_k$ form part of a system of parameters for R, and $u^p \in (x_1^p, \ldots, x_k^p)R$ implies that $u \in (x_1, \ldots, x_k)R \cap A = (x_1, \ldots, x_k)A$, as required, since R is faithfully flat over A. he $\square$

We can now prove:

Theorem. *Let R be a Noetherian ring of positive prime characteristic p, and suppose either that (R, m, K) is local or that R is finitely generated $\mathbb{N}$-graded over $R_0 = K$, a field, and that m is the homogeneous maximal ideal. Let $I \subseteq m$ be an ideal. Let $\mathcal{M}$ be the maximal ideal of $\mathrm{gr}_I R$ that is the kernel of the composite surjection $\mathrm{gr}_I R \twoheadrightarrow R/I \twoheadrightarrow R/m$, and suppose that $(\mathrm{gr}_I R)_{\mathcal{M}}$ is Cohen-Macaulay F-injective. Then R is Cohen-Macaulay F-injective.*

Proof. The argument is quite similar to the one given for the Cohen-Macaulay and Gorenstein properties in the Theorem on page 3 of the Lecture Notes for September 28. One forms the second Rees ring $S = R[It, v]$, which maps onto $S/vS \cong \mathrm{gr}_I R$, and localizes at the contraction of $\mathcal{M}$, which we call Q. Then $S_Q/(v) \cong (\mathrm{gr}_I R)_{\mathcal{M}}$ is F-injective Cohen-Macaulay, and so S_Q is as well. Let $P \subseteq Q$ be the prime described in the proof of the Theorem on page 3 of the Lecture Notes of September 28. By part (a) of the Proposition above, $S_P \cong R(t)$ is Cohen-Macaulay F-injective, since it is a localization of S_Q. Hence R is Cohen-Macaulay F-injective, by part (b) of the Proposition above. $\square$

Corollary. *Let R be a Hodge algebra over a field K of characteristic $p > 0$, and suppose that the corresponding discrete Hodge algebra is Cohen-Macaulay and reduced: the condition that the discrete Hodge algebra be reduced holds whenever R is an ASL. Then R is Cohen-Macaulay and F-injective.*

Proof. When it is reduced, the corresponding discrete Hodge algebra is a face ring, and we have seen that face rings over a field are F-split and therefore F-injective in characteristic

$p > 0$. The result now follows from the Theorem above, and the fact that there is a sequence of associated graded rings from R to the corresponding discrete Hodge algebra. $\square$

Let $X = (x_{ij})$ be an $r \times s$ matrix of indeterminates, where $1 \le r \le s$, over a base ring K, and let $K[X/r]$ be the subring of the polynomial ring $K[X]$ in the indeterminates generated by the $r \times r$ minors of X. As mentioned earlier, this is the homogeneous coordinate ring of the Grassmann variety of r-dimensional subspaces of K^s. We want to prove that this ring is an ASL on the poset H of minors. We shall write $X[a_1, \ldots, a_r]$ for the determinant of the matrix formed from the columns of X indexed by the integers $a_1, \ldots, a_r$, which are required to be integers satisfying $1 \le a_j \le s$. In the standard description of a minor we shall assume that $a_1 < a_2 < \cdots < a_r$. However, the symbol $X[a_1, \ldots, a_r]$ has meaning in any case: if $a_j = a_k$ for $j \ne k$, then $X[a_1, \ldots, a_r] = 0$, and if π is a permutation of the integers $1, \ldots, r$, then $X[a_{\pi(1)}, a_{\pi(2)}, \ldots, a_{\pi(r)}] = \mathrm{sgn}\,(\pi) X[a_1, \ldots, a_r]$, where $\mathrm{sgn}\,(\pi) \in \{\pm 1\}$ is the sign of the permutation π. Recall that H is partiall ordered so that when $a_1 < a_2 < \cdots < a_r$ and $b_1 < b_2 < \cdots < b_r$, $X[a_1, \ldots, a_r] \le X[b_1, \ldots, b_r]$ means that $a_j \le b_j$ for $1 \le j \le r$. The standard monomials are those such that the set of minors occurring is linearly ordered.

We first want to show that the standard monomials are linearly independent over K. In order to prove this, we introduce several matrices Y_h, one for each element $h \in H$, the poset of minors. Specifically, let $Y = (y_{ij})$ be a matrix of indeterminates, and suppose that we are given $h \in H$, say $h = X[a_1, \ldots, a_r]$ where $a_1 < a_2 < \cdots < a_r$. We define Y_h to be the matrix obtained from Y by replacing the $a_i - 1$ leftmost variables $y_{i1}, \ldots, y_{i,a_i-1}$ of the $i\,th$ row by 0, while leaving all other entries of the ith row unchanged. Then there is a K-algebra homomorphism $K[X] \to K[Y_h]$ that maps each entry of X to the corresponding entry of Y_h: $x_{ij} \mapsto 0$ if $j < a_i$, and $x_{ij} \mapsto y_{ij}$ if $j \ge a_i$. This map restricts to a map $\theta_h : K[X/r] \to K[Y_h/r]$. Also note that if $h \le h'$, where $h' = X[b_1, \ldots, b_r]$ with $b_1 < \cdots < b_r$, then there is a K-alghebra map $K[Y_h] \to K[Y'_h]$ that sends $y_{ij} \mapsto 0$ if $a_i \le j < b_i$ and $y_{ij} \mapsto y_{ij}$ if $j \ge b_i$. Again, this induces a K-algebra homomorphism $\lambda_{h,h'} : K[Y_h] \to K[Y_{h'}]$ when $h \le h'$, and it is clear that $\lambda_{h,h'} \circ \theta_h = \theta'_h$.

Let $\mathcal{M}_h$ denote the set of standard monomials that are $\ge h$. We shall prove that for all $h \in H$, the elements $\{\theta_h(\mu) : \mu \in \mathcal{M}_h\}$ is a K-linearly independent set indexed by $\mathcal{M}_h$. If we take $h_0 = X[1, \ldots, r]$, the minimum element of H, we find that the images of the standard monomials under θ_{h_0} are linearly independent over K, and it follows that the standard monomials themselves are linearly independent over K.

We first note that θ_h has the following critical property:

$(**)$ θ_h kills every minor $h' = X[b_1, \ldots, b_r]$ with $b_1 < \cdots <, b_r$ such that h' is *not* $\ge h$.

The reason is that for some i, we have that $b_i < a_i$. This implies that the ith row of the matrix consisting of the columns of Y_h indexed by $b_1, \ldots, b_i$ is 0, and so this matrix, which has i columns, has rank $\le i-1$. But then the $r-i$ additional columns indexed $b_{i+1}, \ldots, b_r$ can increase the rank at most to $i - 1 + (r - i) = r - 1$, and so $Y_h[b_1, \ldots, b_r] = 0$.

To prove the result, we use a sort of reverse induction on h. Choose h maximal in H for which the result is false, and suppose there is nonzero K-relation on the images of certain standard monomials $\mu_1, \ldots, \mu_n$: we may take these of smallest possible degree, and we may assume that every μ_j occurs with nonzero coefficient.

We consider two cases. The first case is that each of the μ_j has h as a factor and can be written $h\nu_j$. Note that $\theta_h(h) = y_{a_1} \cdots y_{a_r}$ is not a zerodivisor in $K[Y_h]$, nor in $K[Y_h/r]$. It follows that we get a K-relation on the elements $\theta_h(\nu_j)$, and the degrees have decreased.

Therefore we may assume that there is at least one element μ' that is not divisible by h: call its smallest factor h'. We now apply $\lambda_{h,h'}$ to this relation. This has the same effect as applying $\theta_{h'}$ to the original relation. This does not kill the term in the linear combination that is the image of a multiple of μ' with nonzero coefficient from K, but it does kill all terms that involve an element $h'' \in H$ that is not $\geq h'$ by property $(**)$ proved above. This gives a nonzero relation on elements that are in the image of $\mathcal{M}_{h'}$ under $\theta_{h'}$, a contradiction. $\square$

Our next objective is to describe the Plücker relations on the minors of a matrix. We assume that we are given nonnegative integers a, t, u, b such that $a+t = r$, $u+b = r$, t, $u >$ 0, and $t+u = m > r$. We also assume given indices $i_1, \ldots, i_a$, $j_1, \ldots, j_m$, $k_1, \ldots, k_b$. Let $\mathcal{N}$ denote the set of permutations ν of $1, \ldots, m$ such that, writing ν_c for $\nu(c)$, we have $j_{\nu_1} < \cdots < j_{\nu_t}$ and $j_{\nu_{t+1}} < \cdots < j_{\nu_m}$. Then

$$\sum_{\nu \in \mathcal{N}} \operatorname{sgn}(\nu) X[i_1, \ldots, i_a, j_{\nu_1}, \ldots j_{\nu_t}] X[j_{\nu_{t+1}}, \ldots, j_{\nu_m}, k_1, \ldots, k_b] = 0.$$

This is a typical Plücker relation. We shall prove the validity of these determinantal identities, and then show that they suffice to give straightening relations for $K[X/r]$. Note that in order to prove these relations, it suffices to do the case where the entries of the matrix X are indeterminates over $\mathbb{Z}$, and then we may pass to the field of fractions $\mathbb{Q}(X)$ of $\mathbb{Z}[X]$. Therefore, it suffices to prove that these identities when the matrix has entries in a field L of characteristic 0.