Math 711: Lecture of October 21, 2005

A poset is called *pure* if any two maximal chains have the same length.

Proposition. *Let H be a finite poset.*

(a) *If H is a distributive lattice then H is locally upper semimodular.*

(b) *If H is locally upper semimodular and H_0 is a subposet such that whenever $x \in H_0$, $y \in H$ and $x \leq y$, then $y \in H_0$, then H_0 is locally upper semimodular.*

(c) *If H is locally upper semimodular then the subposet consisting of all elements $\leq x \in H$ is locally upper semimodular.*

(d) *If H is locally upper semimodular and $x \leq y$ are elements of H, then the subposet $\{z \in H : x \leq z \leq y\}$ is bounded and locally upper semimodular.*

(e) *If H is bounded and locally upper semimodular then H is pure.*

(f) *If H is a locally upper semimodular and $x \leq y$ in H, then any two saturated chains joining x to y have the same length.*

Proof. For (a), suppose that H is a distributive lattice and that y, z are covers of x, and y, $z < v$. If $y = z$ any element $\leq v$ minimal with respect to being $> x$ is a common cover. If $x \neq y$, we shall show that $w = y \vee z$ is a cover of y: it is a cover of z by symmetry. Evidently, $w > y$. Suppose $w > h > y$. Then $h = h \wedge w = h \ \& \ (y \vee z) = (h \wedge y) \vee (h \wedge z) = y \vee (h \wedge z)$, and so it suffices to see that $h \wedge z = x$. Clearly, $x \leq h \wedge z \wedge z$, and z is a cover of x, $h \wedge z$ is either x or z. In the second case, $h \geq y$ and $h \geq z$ implies $h \geq w$, so we are done in either case.

Parts (b) and (c) are immediate from the definition, and together they imply (d). To prove (e), we use induction on the size of the poset. Consider any two maximal chains of supposedly different length, which must go from the least element, call it 0, to the greatest element, call it 1. Suppose one of the chains, of length L, has y as its smallest nonzero term and the other, of length L', has z as its smallest nonzero term. If $y = z$ we get a smaller counterexample in the poset of elements between y and 1. If not, let w be a common cover, and choose any saturated chain C from w to 1. Then y together with C gives a saturated chain from y to 1, which, by the induction hypothesis, has the same length $L - 1$ as our original chain with its first term, 0, deleted. But, by the same reasoning, z together with C has length $L' - 1$, and so $L - 1 = L' - 1$.

Part (f) is immediate from part (e). $\square$

If H is a poset we may want to introduce a new element, which we denote $\widehat{0}$, to serve as a least element, and/or a new element, which we denote $\widehat{1}$, to serve as a greatest element.

Lemma. *Let H be a finite poset with a greatest element. Then $H \cup \{\widehat{0}\}$ is locally upper semimodular if and only if H is locally upper semimodular and for any two minimal elements x, y of H with x, $y < v$, there is a common cover w of x and y with $w \leq v$.*

Proof. The fact that if $H \cup \{\widehat{0}\}$ is locally upper semimodular then H is locally upper semimodular follows from part (b) of the Proposition just above. The condition on pairs of minimal elements of H must hold if $H \cup \{\widehat{0}\}$ is locally upper semimodular because any two are covers of $\widehat{0}$ in $H \cup \{\widehat{0}\}$. The "if" part is straightforward from the definition. $\square$

Lemma. *Let H be a poset. Let Δ_H be the order complex of H. Let H' be the union of H and a subset of $\{\widehat{0}, \widehat{1}\}$. Then, for any ring K, $K[\Delta_H]$ is Cohen-Macaulay if and only if $K[\Delta_{H'}]$ is Cohen-Macaulay. In fact, $K[\Delta_{H'}]$ is a polynomial ring over $K[\Delta_H]$ in variables corresponding to the elements of $H' - H$.*

Proof. Because $\widehat{0}$ and $\widehat{1}$ (whichever are present) are comparable to every element of H', they do not kill any monomial of $K[\Delta_H]$ or $K[\Delta_{H'}]$, from which the final statement is clear. $\square$

Theorem. *Let H be a finite poset with order complex $\Delta = \Delta_H$, and let K be a Cohen-Macaulay ring. Then if*

(1) *H is bounded and is locally upper semimodular or*
(2) *H has a greatest element and $H \cup \{\widehat{0}\}$ is locally upper semimodular or*
(3) *$H \cup \{\widehat{0}, \widehat{1}\}$ is locally upper semimodular*

then $K[\Delta]$ is Cohen-Macaulay.

Proof. It suffices to do the case where K is a field.

We shall prove (1) and (2) (statement (1) for H is equivalent to statement (2) for $H - \{h_0\}$, where h_0 is the least element of H, by the preceding Lemma). Note that (3) follows from (1) by the preceding Lemma. We use induction on the cardinality of H. If H has a minimum element h_0 we pass to statement (2) for $H - \{h_0\}$. If this poset has a least element we are in the case of statement (1) again and may repeat this step.

Therefore, we may assume that $H_1 = H - \{h_0\}$ has several minimal elements. H_1 is pure: its maximal chains are all one less in length than a maximal chains of H. The minimal elements of H_1 are the height one elements of H, and these are the same as the covers of h_0. Let $R = K[\Delta_{H_1}]$. Choose one of the minimal elements: call it h_1. Let T be the set of minimal elements of H_1 other than h_1. Let $I \subseteq R$ be the ideal generated by the set S of elements $h \in H_1$ such that $h \geq h_1$ but is not comparable to any other minimal element. Let $J \subseteq R$ be generated by the set T' of elements of H_1 incomparable to h_1: note that $T \subseteq T'$. Then R/I is the face ring over K of the order complex of $H_1 - S$, whose minimal elements are simply the elements of the set T, and which is locally upper semimodular by part (b) of the first Proposition. Call the dimension of this ring n: it is the same as the dimension of R. Then R/I is Cohen-Macaulay by the induction hypothesis. Moreover, R/J is the face ring over K of the order complex of $H_1 - T'$: this is the set of elements $\geq h_1$, and is bounded and locally upper semimodular giving another ring of dimension n that is Cohen-Macaulay.

Finally, consider $R/(I + J)$. The elements of H_1 that remain are those that are $\geq h_1$ and also $\geq$ some element of T. For any such v we have $v \geq w$ where w is a common cover of h_1 and an element of T. It follows that the common covers C of h_1 and another element of T are the minimal elements of the poset $H_1 - S - T'$, and the latter can be described as consisting of all elements $\geq$ an element of C. Because any two of these minimal elements are covers of h_1, it follows that $H_1 - S - T$ satisfies the condition in (2), and so $R/(I + J)$ is Cohen-Macaulay of dimension $n - 1$. Because all elements of T' are incomparable to all elements of S, $I \cap J = 0$. Therefore there is a short exact sequence

$$0 \to R \to R/I \oplus R/J \to R/(I + J) \to 0$$

where the middle term is Cohen-Macaulay and has depth n on the homogeneous maximal ideal of R, the rightmost term is Cohen-Macaulay and has dimension $n-1$, and the leftmost term has dimension n. It follows as in the argument in the middle of page 2 of the Lecture Notes of September 12 or on page 1 of the Lecture Notes of September 19 that R has depth n on its homogenous maximal ideal, and so is Cohen-Macaulay. $\square$

We have at once:

Theorem. *Let R be an ASL over K on H where H satisfies on of the three conditions of the preceding Theorem. In particular, it suffices if H is a distributive lattice. If K is Cohen-Macaulay, then R is Cohen-Macaulay.* $\square$

Since we have already seen that the $r \times r$ minors form a distributive lattice, we have:

Theorem. *Let X be an $r \times s$ matrix of indeterminates, $1 \leq r \leq s$, over a Cohen-Macaulay ring K. Then $K[X/r]$ is Cohen-Macaulay.* $\square$