Math 711: Lecture of October 24, 2005

Let X be an $r \times s$ matrix of indeterminates over a base ring K. Here $1 \le r$ and $1 \le s$, but we do not need to assume that $r \le s$. We want to put a "non-standard" ASL structure on the polynomial ring $K[X]$. To this end, let Z be an $r \times r$ matrix of new indeterminates, and form the concatenation $Y = [X \,|\, Z]$. We shall let $x_{i,j}$ denote the typical entry of Y. Let Π be the permuation matrix whose columns $e_r, \, \ldots \, , e_1$ are the elements of the standard basis for K^r in reverse order. Then there is a unique K-homomorphism $K[Y] \to K[X]$ such that the image of the matrix Y is the matrix $\overline{Y} = [X \,|\, \Pi]$. Thus, each entry of X is fixed while Z is specialized to the matrix Π. Let $\det(\Pi) = \iota$, which will be ± 1.

This homomorphism induces a surjective homomorphism $K[Y/r] \twoheadrightarrow K[\overline{Y}/r]$. Note that the entry $x_{i,j}$ of X is, up to sign, the determinant of the $r \times r$ submatrix of $\overline{Y}$ whose columns are the j th column of Y (which is the same as the j th column of X and all the columns of Z except the one that is e_i. It follows that $K[\overline{Y}/r] = K[X]$, and so we have a K-algebra surjection $K[Y/r] \twoheadrightarrow K[X]$.

This map $\theta : K[Y/r] \twoheadrightarrow K[X]$ takes H, the poset of $r \times r$ minors of Y, bijectively onto $H' \cup \{\pm 1\}$ where H' is the set of $t \times t$ minors of X as t varies, $1 \le t \le \min\{r, \, s\}$, where each minor may need a sign adjustment. If one views 1 as the determinant of a 0×0 submatrix of X, we may think of $H' \cup \{\pm 1\}$ as the set of all minors of X. The point is that an $r \times r$ minor M of Y whose columns include precisely $r - t$ of the final r columns of Y, to wit, the columns that map to $e_{j_1}, \, \ldots \, , e_{j_{r-t}}$, will map to the $t \times t$ minor of X whose columns have the indices of those columns of M that come from X, and whose rows are indexed by the integers in the set

$$\{1, \, \ldots \, , r\} - \{e_{j_1}, \, \ldots \, , e_{j_{r-t}}\}.$$

The largest minor ω of Y maps to $\iota = \pm 1$.

We introduce the following notation for minors of X: $X[i_1, \, \ldots \, , i_t \,|\, j_1, \, \ldots \, , j_t]$ denotes the minor of X formed from the rows numbered with the indices $i_1, \, \ldots \, , i_t$ and the columns with the indices $j_1, \, \ldots \, , j_t$. In a standard description of a minor, we assume that

$$1 \le i_1 < \cdots < i_t < r \text{ and that } 1 \le j_1 < \cdots < j_t \le s.$$

We can partially order H' by the rule that $\mu \le \nu$, where μ has, up to sign, the standard description $X[i_1, \, \ldots \, , i_t \,|\, j_1, \, \ldots \, , j_t]$, and ν has, up to sign, the standard description $X[i'_1, \, \ldots \, , i'_u \,|\, j'_1, \, \ldots \, , j'_u]$, precisely if $t \ge u$, $i_a \le i'_a$, $1 \le a \le u$, and $j_a \le j'_a$, $1 \le a \le u$. It is easy to verify that θ maps $H - \{\omega\}$ order isomorphically onto H' with this order.

Theorem. *Let X be an $r \times s$ matrix of indeterminates over the ring K, where $r \ge 1$ and $s \ge 1$. The $K[X]$ is an ASL on H', the set of all minors of X of sizes 1 through $\min\{r, \, s\}$, where H' has the order described above.*

We need some preliminaries before we give the proof. The idea of the proof is deduce this from the corresponding fact about $K[Y/r]$ and H.

Lemma. *Let $Y = [X \,|\, Z]$ be an $r \times (r+s)$ matrix of indeterminates as above, and let $\omega = \det(Z)$. Let $U = (u_{i,j})$ be an $r \times s$ matrix of new indeterminates. Then $K[Y/r]_\omega \cong K[U][\omega,\, 1/\omega]$, where the $u_{i,j}$ and ω are algebraically independent. The isomorphism is a K-isomorphism that fixes ω and maps the matrix U to the matrix XZ^{-1}.*

Proof. The $r \times r$ minors of the matrix $Y' = [XZ^{-1} \,|\, I] = YZ^{-1}$ are the same as the corresponding minors of Y with each divided multiplied by $\det(Z^{-1}) = 1/\omega$. Therefore $K[Y'/r][\omega,\, 1/\omega] = K[X/r]_\omega$. The entries of XZ^{-1} all occur as $r \times r$ minors of Y': to get the i, j entry of XZ^{-1} (up to sign) one can take the minor of the submatrix formed by the j th column of X and all of the columns of the $r \times r$ identity matrix $I = I_r$ except e_i. Thus, the result will follow if the entries of XZ^{-1} and ω are algebraically independent. This is clear because we can map $K[Y] \to K[U, Z]$ via the map sending X to UZ and Z to Z, and then YZ^{-1} maps to U while $\omega = \det(Z)$. $\square$

This shows again that when K is a field, $\dim\left(K[Y/r]\right) = \dim\left(K[Y_r]_\omega\right) = \dim\left(K[U,\, \omega, 1/\omega\right] = rs + 1$, which agrees with the result determined earlier ($s + r$ replaces s).

We can now show:

Lemma. *The kernel of the map from $\theta : K[Y/r] \twoheadrightarrow K[X]$ described above is the principal ideal $(\omega - \iota)$, where $\iota = \pm 1$ is the image of ω under θ.*

Proof. Evidently, the specified ideal is in the kernel, and we have a surjection

$$K[Y/r]/(\omega - \iota) \twoheadrightarrow K[X]$$

that we must prove is injective. Since the image of ω is invertible on both sides, we may localize at ω without affecting whether this map is an isomorphism, and so we may study instead the map

$$K[Y/r]_\omega/(\omega - \iota) \twoheadrightarrow K[X].$$

The result is now obvious when we make use of the fact that

$$K[Y/r][\omega,\, \omega^{-1}] \cong K[U,\, \omega, \omega^{-1}]$$

as described in the preceding Lemma. $\square$

Proof of the Theorem. We first show that the standard monomials in the elements of H' are linearly independent over K. Each such standard monomial lifts in an obvious way to a standard monomial of $K[Y/r]$ that does not involve ω. Therefore, if independence fails, we get an element $\sum_\mu \lambda_\mu \mu$ that is in the kernel, where the λ_μ are nonzero elements of K, the set of μ that occur is nonempty, and and no μ involves ω. This leads to an equation

$$\sum_\mu \lambda_\mu \mu = (\omega - \iota)(\sum_\nu \lambda'_\nu \nu)$$

where the set of ν that occur is non-empty, each λ'_ν is a nonzero element of K, and the ν are standard monomials that may involve ω. Choose ν_1 occurring in the sum on the right hand side so that the degree with which ω occurs in ν_1 is maximum (it may be zero). When one uses the distributive law to expand the right hand side, all the products are nonzero scalars times standard monomials, and the term involving $\nu_1\omega$ cannot be canceled. But there is no term involving ω on the right.

It remains to show that there is a straightening law. Given two incomparable elements of H', we may lift them to H: thus, we may think of starting with two incomparable elements h, h' of H, neither of which is ω. The straightening law for $K[Y/r]$ gives a relation

$$(*) \quad hh' = \sum_t \lambda_t h_t^{(1)} h_t^{(2)}$$

where the $\lambda_t \in K$, and in each term $h_t^{(1)} \leq h_t^{(2)}$ and $h_t^{(1)} < h,\, h'$. In particular, $h_t^{(1)}$ is never ω. When we apply θ, the only change is that in those terms where $h_t^{(2)} = \omega$, $h_t^{(2)}$ is replaced by ι, while $h_t^{(1)}$ persists. This means that the image of the relation $(*)$ gives a straightening relation for the images of h and h' under θ.

Note that the elements of H' are all homogeneous of positive degree, so that $K[X]$ is appropriately graded, although this grading is not induced by the grading on $K[Y/r]$. $\square$

A subset H_0 of a poset H is called a (poset) *ideal* if whenever $h \in H$, $h_0 \in H_0$ and $h \leq h_0$, then $h \in H_0$. The complement of an ideal is called a *co-ideal.* $H_!$ is a co-ideal if and only if whenever $h_1 \in H_1$ and $h \in H$ with $h \geq h_1$, then $h \in H_1$. We note the following easy but important fact:

Proposition. *Let R be an ASL over K on H. Let $H_0 \subseteq H$ be a poset ideal. Then $(H_0)R$ is the K-span of those standard monomials containing a factor from H_0, and $R/(H_0)R$ is an ASL over K on $H_1 = H - H_0$.*

Proof. Let J be the K-span of the standard monomials containing a factor from H_0. It suffices to prove that for every element $h_0 \in H_0$, $h_0R \subseteq J$, for J is closed under addition, and so this will yield that $(H_0)R \subseteq J$. The other inclusion is obvious.

If some h_0R is not contained in J, we can choose h_0 minimal with respect to this property, and then the product of h_0 with some monomial μ in the elements of H must fail to be in J. Then $h_0\mu$ cannot be standard, and so h_0 is incomparable to some factor h of μ: say $\mu = h\nu$. Then $h_0h\nu \notin J$. But h_0h is a linear combination of standard monomials, each with a factor that is strictly less than h_0 and necessarily in H_0. The result now follows from the minimality of h_0. $\square$

Now consider the poset H' of all minors of X. Consider the ideal of elements that are not $\geq X[1, \ldots, t \,|\, 1, \ldots, t]$. This ideal contains all minors that are of size $t+1$ or greater. If we reintroduce a greatest element in the complementary coideal, it is a distributive lattice.

Therefore:

Theorem. *With K and X as above, $K[X]/I_{t+1}(X)$ is an ASL over K on a poset H (all minors of size t or smaller) such that $H \cup \{\widehat{1}\}$ is a distributive lattice. Therefore, $K[X]/I_{t+1}$ is a Cohen-Macaulay ring whenever K is Cohen-Macaulay, and is reduced whenever K is reduced.*