Math 711: Lecture of October 26, 2005

We have seen, using Hodge algebra techniques, that $K[X]/I_t(X)$ is a Cohen-Macaulay reduced ring. First consider the case where K is algebraically closed. We know that the algebraic set defined by $I_t(X)$ is irreducible, and so these rings are domains. This implies the result for any field, since tensoring over the field K with its algebraic closure $\overline{K}$ will produce an extension ring that is a domain. We want to prove more. Recall that we earlier showed that the dimension of this ring is $(t-1)\bigl((r+s-(t-1)\bigr)$, and so the height of I is $rs-(t-1)\bigl((r+s-(t-1)\bigr) = \bigl(r-(t-1)\bigr)\bigl(s-(t-1)\bigr)$.

Theorem. *Let X be an $r \times s$ matrix of indeterminates over the field K, and suppose that $1 \le t \le \min\{r, s\}$. The ring $K[X]/I_t(X)$ is regular precisely when $t = 1$. It is a normal Cohen-Macaulay domain of dimension $(t-1)\bigl((r+s-(t-1)\bigr)$, and the singular locus is defined by the image of $I_{t-1}(X)$ (which we define to be the unit ideal when $t = 1$).*

Proof. When $t \ge 2$ the ring is clearly not regular when localized at its homogeneous maximal ideal, since the defining equations are in the square of the maximal ideal.

We want to study what happens to $R = K[X]/I_t(X)$ when we localize at a $t-1$ size minor. Since we may permute rows and columns, it does not matter which minor we choose, and we let M denote the $(t-1) \times (t-1)$ submatrix in the upper left hand corner of X. Let $D = \det(M)$. We first want to give a description of $K[X]_D$. Think of X as having the block form

$$\begin{pmatrix} M & Y \\ Z & U \end{pmatrix}$$

where the blocks M, Y, Z, U have sizes $(t-1)\times(t-1)$, $(t-1)\times\bigl(s-(t-1)\bigr)$, $\bigl(r-(t-1)\bigr)\times(t-1)$, and $\bigl(r-(t-1)\bigr) \times \bigl(s-(t-1)\bigr)$ respectively. We can multiply X by the $r \times r$ block matrix

$$\begin{pmatrix} M^{-1} & 0 \\ 0 & I \end{pmatrix}$$

to obtain

$$\begin{pmatrix} I & M^{-1}Y \\ Z & U \end{pmatrix}.$$

We can think of $I_t(X)$ as the ideal generated by the size t minors of this matrix.

If we think of $Y' = M^{-1}Y$ as a new matrix of indeterminates, we see that $K[X]_D = K[M, Y, Z, U]_D = K[M, Y', Z, U]_D$, where the entries of M, Y', Z, and U must be algebraically independent. We think of $I_t(X)$ as

$$I_t\Bigl(\begin{pmatrix} I & Y' \\ Z & U \end{pmatrix}\Bigr).$$

We now multiply by the invertible $r \times r$ matrix whose blolck form is $\begin{pmatrix} I & 0 \\ -Z & I \end{pmatrix}$, getting $\begin{pmatrix} I & Y' \\ 0 & U - ZY' \end{pmatrix}$. If we write $U' = U - ZY'$ then we see that $K[X]_D = K[M, Y', Z, U']_D$, where the entries of M, Y', Z, and U' are algebraically independent over K. Each $u'_{i,j}$ is obtained as the $t \times t$ minor gotten by using its row and column along with the rows and columns of the $(t-1) \times (t-1)$ identity matrix in the upper left corner. Moreover, it is clear that the vanishing of the $u'_{i,j}$ forces all $t \times t$ minors to vanish, since all but the first $t-1$ rows of the matrix becomes 0. Thus, $K[X]_D/I_t(X) \cong K[M, Y', Z, U']_D/I_1(U') \cong K[M, Y', Z]_D$, as required.

This establishes our claim: in particular, localizing at any $t-1$ size minor produces a regular ring. It follows that the defining ideal of the singular locus must contain the image of $I_{t-1}(X)$.

We next want to see that the ring remains singular when localized at the prime ideal P generated by the $t-1$ size minors, where we assume that $t \geq 2$. If $t = 2$, this is clear, since the generators of the ideal are in the square of the maximal ideal after localization. We assume $t \geq 2$ and we use induction on t. Suppose we only adjoin the inverse of $x_{1,1}$. Then we may multiply the first row by the inverse of $x_{1,1}$, and perform elementary row and column operations to make the rest of the first row and column 0. The matrix now has the block form

$$\begin{pmatrix} 1 & 0 \\ 0 & U \end{pmatrix}$$

where the matrix on the upper left is simply (1), and $U = (u_{i,j})$ is $(r-1) \times (s-1)$. Let $x = x_{1,1}$. Then $K[X]_x \cong K[Y, U]_x$ where $Y = x_{1,1}, \ldots x_{1,s}, x_{2,1}, \ldots, x_{r,1}$ and the $u_{i,j}$ are algebraically independent, and

$$I_t(X)K[X]_x = I_t\left(\begin{pmatrix} 1 & 0 \\ 0 & U \end{pmatrix}\right)K[Y, U]_x = I_{t-1}(U),$$

while P expands to $I_{t-2}(U)$ in $K[Y, U]_x$. The result now follows from the induction hypothesis.

To prove normality we may assume that $t \geq 2$ (otherwise the ring is regular), and it suffices to prove that the defining ideal J of the singular locus has depth ≥ 2. (This immediately implies that the local ring at a height one prime is regular, and so a DVR, while principal ideals are unmixed: if P is an associated prime of a principal ideal and has height ≥ 2, R_P has depth 1 and is not regular, a contradiction, since $P \supseteq J$ and J has depth at least 2.) Since the ring is Cohen-Macaulay, its depth is the same as its height, and this may be calculated as the difference of the dimension of the two rings obtained by killing the ideals of minors, which is

$$(t-1)(r+s)-(t-1)^2-\big((t-2)(r+s)-(t-2)^2\big) = r+s-\big((t-1)^2-(t-2)^2\big) = r+s-2t+3.$$

This is $(r-t)+(s-t)+3 \geq 3$, which is more than we need. $\square$

Discussion: divisor class groups of normal domains. The divisor class group is discussed in the Math 614 Fall 2003 Leccture Notes, Lecture of December 1, and in the Math 615 Winter 2004 Lecture Notes, Lectures of March 29 and March 31.

We recall some basic facts about the divisor class group $\mathcal{C}\ell\,(R)$ of a normal Noetherian domain R. This group may be thought of as a quotient of the free abelian group with generators $[P]$ corresponding to the height one prime ideals P of R. The relations are the span of the elements $\mathrm{div}\,(r)$, where $r \in R - \{0\}$: the primary decomposition of rR will have the form $P_1^{(n_1)} \cap \cdots \cap P_k^{(n_k)}$, where the P_j are the mutually distinct height one primes of rR, and then

$$\mathrm{div}\,(r) = \sum_{j=1}^{k} n_j[P_j].$$

The sum may be thought of instead as extended over all height one primes of R, where the coefficient of P is the order of r in the discrete valuation ring R_P. For each r, all but finitely many coefficients are 0. Note that $\mathrm{div}\,(1) = 0$, since the set of associated primes of the ideal R is empty. If I is a nonzero ideal all of whose associated primes are of height one, it will also have a primary decomposition of the form $P_1^{(n_1)} \cap \cdots \cap P_k^{(n_k)}$, and we define

$$\mathrm{div}\,(I) = \sum_{j=1}^{k} n_j[P_j].$$

Then $\mathrm{div}\,(r) = \mathrm{div}\,(rR)$.

An alternative point of view is that the elements of $\mathcal{C}\ell\,(R)$ are in bijective correspondence with the rank one reflexive modules over R . These are torsion-free, and can be embedded in R. Each is therefore isomorphic to a nonzero ideal of R, where the ideal is thought of as an R-module. A nonzero ideal is reflexive as an R-module if and only if its primary decomposition involves only height one primes (we view this as vacuously true of R), and we let the isomorphism class of I correspond to the class of $\mathrm{div}\,(I)$ to get the isomorphism with the divisor class group as defined in the paragraph above. From the module point of view, the product of the classes of M and N is the double dual into R of $M \otimes_R N$, while the inverse of the class represented by M is represented by $\mathrm{Hom}_R(M,\, R)$.

R is a UFD if and only if its class group is trivial, which says that every rank one reflexive module is isomorphic with R.

We next note the following:

Theorem. *Let R be a normal Noetherian domain and let W be a multiplicative system in R. Then there is a surjective map $\mathcal{C}\ell\,(R) \to \mathcal{C}\ell\,(W^{-1}R)$ that sends $[P]$ in R to $[PW^{-1}R]$ if P is disjoint from W and to 0 if P meets W. The kernel of this map is spanned by $\{[P] : P \cap W \neq \emptyset\}$. In particular, if W is generated by prime elements of R, we have an isomorphism $\mathcal{C}\ell\,(R) \cong \mathcal{C}\ell\,(W^{-1}R)$.*

Proof. The height one primes of R not meeting W map bijectively onto the height one primes of $W^{-1}R$. Since every nonzero element of $W^{-1}R$ is a unit times an element of R, in calculating $\mathcal{C}\ell\,(W^{-1}R)$ we need only kill the span of the elements $\mathrm{div}\,(r)$ for $r \in R-\{0\}$. But $\mathrm{div}\,(r)$ over $W^{-1}R$ is the same as $\mathrm{div}\,(r)$ over R with the height one primes that meet W omitted.

The final statement follows because the classes of principal primes are already zero in $\mathcal{C}\ell\,(R)$. □

We want to use this fact to show that $K[X/r]$ is a UFD. We first note:

Proposition. *Let R be an ASL over a ring K on a bounded poset H with least element h_0 and greatest element h_1, where $h_0 < h_1$. Then h_0, h_1 is a permutable regular sequence in R.*

Proof. Both are nonzerodivisors, because when we multiply a nonzero linear combination of standard monomials by either, we get a nonzero linear combination of standard monomials (with the same number of nonzero terms). By an earlier lemma, h_0R is the span of the standard monomials involving h_0. Suppose that h_0 divides h_1r. When r is written as a linear combination of standard monomials, each must involve h_0: if one does not, that will remain true after we multiply by h_1. Permutability follows from Nakayama's lemma, since we are in the nonnegatively graded case. □

Theorem. *Let X be an $r \times s$ matrix over a Noetherian UFD K, where $1 \leq r \leq s$. Then $K[X/r]$ is a UFD.*

Proof. If $s = r$ then $K[X/r]$ is a polynomial ring in one variable over K and so is a UFD. To prove the result when $s > r$ we shall show that the smallest minor $\alpha = X[1, \ldots, r]$ of X is prime. We already know that $K[X/r]_\alpha$ is a localization of a polynomial ring over K (we proved this for the largest minor, ω, but it clearly does not matter which minor we use), and has trivial divisor class group. By the Theorem above on the effect of localization on divisor class groups, it follows that the divisor class group of $K[X/r]$ is trivial as well.

To show that $K[X/r]/(\alpha)$ is a domain, we first note that α, ω is a regular sequence in $K[x/r]$, so that ω is not a zerodivisor on $K[X/r]/(\alpha)$. Therefore,

$$K[X/r]/(\alpha) \subseteq \big(K[X/r]/(\alpha)\big)_\omega \cong K[X/r]_\omega)/(\alpha),$$

and so it suffices to show that the image of α in $K[X/r]_\omega \cong K[U][\omega, 1/\omega]$ is prime element of this ring, where U is an $r \times (s - r)$ matrix of indeterminates as in the Lemma at the top of the second page of the Lecture Notes of October 24. Up to multiplication by a unit, the image is the same as a minor, possibly of smaller size than r, of the matrix of indeterminates U, and since all such minors are irreducible in the polynomial ring, we are done. □

Theorem (M. P. Murthy). *Let R be a Cohen-Macaulay ring that is a homomorphic image of a regular ring. Suppose that the local rings of R are UFDs. Then R is Gorenstein.*

Proof. Say that we have $S \twoheadrightarrow R$ where S is regular. Let P be a prime ideal of R and Q its inverse image in S. Then we also have $S_Q \twoheadrightarrow R_P$, and so we may assume that $S \twoheadrightarrow R$ is local. Now consider the Ext dual M of R over S. Since M is Cohen-Macaulay of the same dimension as R, it is torsion-free over the domain R. Since the construction of M commutes with localization, when we tensor with the fraction field of R we obtain the Ext dual of a field, which is a field. Therefore, M has torsion free rank one. Since M is Cohen-Macaulay, it is S_2, and therefore reflexive. Therefore, M represents an element of $\mathcal{C}\ell(R)$, which is trivial. Therefore, $M \cong R$, which shows that R is type 1. □

Remark. It suffices that S be Gorenstein rather than regular in the local surjection $S \twoheadrightarrow R$: if d is the difference of the dimensions, then $M = \mathrm{Ext}^d_S(R, S)$ is a *canonical module* for R in the sense of local cohomology theory. (This will agree with the Ext dual if there is a regular ring S that maps onto R.) The proof can be carried through in identical fashion with this construction of M: this version of M has all of the needed properties.

Corollary. *If K is Gorenstein and X is an $r \times s$ matrix of indeterminates with $1 \le r \le s$, then $K[X/r]$ is Gorenstein.*

Proof. This reduces to the field case. But then we know that $K[X/r]$ is a Cohen-Macaulay UFD that is a homomorphic image of a regular ring, and is therefore Gorenstein. □

Discussion. If K is a field and X is an $r \times s$ matrix of indeterminates over X, and $1 \le t \le r$, then $K[X]/I_t(X)$ is Gorenstein if and only if $t = 1$ or $r = s$. More generally, it is known that the Ext dual M can be calculated as follows. If X_0 is a submatrix consisting of $t-1$ columns of X, then $M \cong \bigl(I_{t-1}(X_0)\bigr)^{s-r}$. When $t = 1$ or $r = s$ this is the unit ideal. For the moment we assume this result.

Assuming it, we note that if X' is the $s \times s$ matrix of indeterminates (x_{ij}), so that X is the submatrix of X' consisting of its first r rows, then $K[X]/I_t(X)$ is an algebra retract of $K[X']/\bigl(I_t(X')\bigr)$. The maps are induced by the inclusion $K[X] \subseteq K[X']$ and the surjection $K[X'] \twoheadrightarrow K[X]$ that fixes $x_{i,j}$ if $i \le r$ and kills $x_{i,j}$ if $i > r$. Each map carries the appropriate denominator ideal into the other, and the compostion

$$K[X] \subseteq K[X'] \twoheadrightarrow K[X]$$

is the identity. It follows for the induced maps that the composition

$$K[X]/I_t(X) \to K[X']/I_t(X') \to K[X]/I_t(X)$$

is the identity. Since an algebra retract of an F-regular ring is F-regular, the problem of showing that the rings $K[X]/I_t(X)$ are F-regular reduces to the case where $r = s$, so that we may assume that the ring is Gorenstein.

We are working towards proving that the rings $K[X/r]$ and $K[X]/I_t(X)$ are F-regular. In both instances, we may assume that we are working in the Gorenstein case. In giving the argument it suffices to consider the case where K is perfect or even algebraically closed, since $\overline{K} \otimes_K R$ is faithfully flat over R.

There are three ingredients in the proof. One shows first, either directly or by induction, that the result holds when one localizes at any homogeneous element of the maximal ideal. Second, one establishes F-injectivity, which we shall prove implies that the ring is actually F-spit in the Gorenstein case. Third, one proves that the $\mathfrak{a}$-invariant, which we shall soon define, is negative. Of course, one must also show that these conditions suffice.

Note that we have already proved that any ASL over a field of characteristic p such that the corresponding discrete Hodge algebra is Cohen-Macaulay is F-injective. See the Corollary at the bottom of the first page of the Lecture Notes of October 14. Hence:

Proposition. *When K is a field of characteristic $p > 0$ and X is an $r \times s$ matrix of indeterminates over K with $1 \le t \le r \le s$, the rings $K[X/r]$ and $K[X]/I_t(X)$ are Cohen-Macaulay and F-injective.* □