Math 711: Lecture of October 28, 2005

A ring of characteristic p is called *F-finite* if $F : R \to R$ is module-finite, i.e., R is a module-finite extension of $F(R) = \{r^p : r \in R\}$, which is also denoted R^p. A finitely generated algebra over an F-finite ring is again F-finite. By induction, this comes down to the case of one algebra generator u, and $1,\, u,\, \dots ,\, u^{p-1}$ span $R[u]$ as a module over $R^p[u^p]$.

Also note that if R has prime characteristic $p > 0$ and is reduced, there is a commutative diagram

$$\begin{array}{ccc} R & \xrightarrow{\subseteq} & R^{1/p} \\ {\scriptstyle \mathrm{id}}\uparrow & & \uparrow{\scriptstyle F^{-1}} \\ R & \xrightarrow[F]{} & R \end{array}$$

where the right hand vertical arrow maps $r \in R$ to $r^{1/p}$. Here, $R^{1/p}$ is the unique reduced ring obtained by adjoining pth roots of all elements of R. (When R is a domain, it is a domain and may be viewed as a subring of the algebraic closure of the fraction field of R. In general, we may inject $R \hookrightarrow \prod_P R/P$ where P runs through the minimal primes of R, and view $R^{1/p}$ as a subring of $\prod_P (R/P)^{1/p}$. $R^{1/p}$ is unique up to unique isomorphism.) When R is F-injective (or F-pure, or F-split), R is reduced. When R is reduced, it is F-split if and only if $R \hookrightarrow R^{1/p}$ splits.

Theorem. *Let R be an $\mathbb{N}$-graded ring finitely generated over R_0, where R_0 is an F-finite (e.g., perfect or algebraically closed) field K of characteristic $p > 0$. If R is F-injective, then R is F-split.*

Proof. We want to show that the short exact sequence

$$0 \to R \to R^{1/p} \to R^{1/p}/R \to 0$$

is split: this is equivalent to the assertion that

$$\mathrm{Hom}_R(R^{1/p},\, R) \to \mathrm{Hom}_R(R,\, R) = R$$

is onto: the element mapping to the identity in $\mathrm{Hom}_R(R,\, R)$ will give the splitting. Let $\mathbb{N} \cdot \frac{1}{p}$ denote the set $\{n/p : n \in \mathbb{N}\}$. Then all of these modules are $(\mathbb{N} \cdot \frac{1}{p})$-graded. It follows that the cokernel C of

$$\mathrm{Hom}_R(R^{1/p},\, R) \to \mathrm{Hom}_R(R,\, R)$$

is also graded. Therefore, if it is not zero, its annihilator in R is a proper homogeneous ideal, and is contained in the homogeneous maximal ideal m of R. We localize at m, and so discover that the cokernel C_m of

$$\mathrm{Hom}_{R_m}\big((R^{1/p})_m,\, R_m\big) \to \mathrm{Hom}_{R_m}(R_m,\, R_m)$$

is also nonzero. Then, since $\widehat{R}$, the completion of R_m, is faithfully flat over R_m, the cokernel remains nonzero when we tensor with $\widehat{R}$ over R. Since $(R^{1/p})_m$ is module-finite over R_m, Hom commutes with flat base change, and so we find that

$$\mathrm{Hom}_{\widehat{R}}(\widehat{R} \otimes_R R^{1/p},\, \widehat{R}) \to \mathrm{Hom}_{\widehat{R}}(\widehat{R},\, \widehat{R})$$

has a nonzero cokernel. Call the new cokernel $\widehat{C}$. We shall get a contradiction.

Let d be the Krull dimension of R. We know that F-injectivity implies that the map $F : H^d_m(R) \to H^d_m(R)$ is injective, and if we identify $F : R \to R$ with $R \subseteq R^{1/p}$, this says that $H^d_m(R) \to H^d_m(R^{1/p})$ is injective. We may identify $H^d_m(R^{1/p})$ with $R^{1/p} \otimes_R H^d_m(R)$, and we note that when R is Gorenstein, $H^d_m(R)$ may be identified with the injective hull E of the residue class field $K = R/m$ of R. Hence, we have an injection $E \to R^{1/p} \otimes_R E$, and applying $\mathrm{Hom}_R(\_\,, E)$ therefore produces a surjection

$$\mathrm{Hom}_R(R^{1/p} \otimes_R E,\, E) \twoheadrightarrow \mathrm{Hom}_R(E,\, E).$$

By the adjointness of tensor and Hom, the left hand term is isomorphic with

$$\mathrm{Hom}_R\big(R^{1/p},\, \mathrm{Hom}_R(E,\, E)\big).$$

Since every element of E is killed by a power of m,

$$\mathrm{Hom}_R(E,\, E) \cong \mathrm{Hom}_{\widehat{R}}(E,\, E) \cong \widehat{R},$$

by Matlis duality. Therefore we have a surjection $\mathrm{Hom}_R(R^{1/p},\, \widehat{R}) \twoheadrightarrow \widehat{R}$. By the universal property of base change, the left hand side may be identified with $\mathrm{Hom}_{\widehat{R}}(\widehat{R} \otimes_R R^{1/p},\, \widehat{R})$. But now we have contradicted the fact that $\widehat{C} \neq 0$. $\square$

We want to calculate the divisor class group of $K[X]/I_t(X)$. We first note:

Proposition. *Let R be a Noetherian normal domain and x an indeterminate over R. Let $S = R[x]$ or $R[x,\, 1/x]$. $\mathcal{C}\ell\,(R) \cong \mathcal{C}\ell\,(S)$ via the map the expands height one primes (or the map that takes a rank one reflexive module M to $S \otimes_R M$).*

Proof. The result for $S = R[x]$ may be deduced from the fact that $\mathcal{C}\ell\,(R)$ may be identified with the quotient of the reduced Grothendieck group $G_0(R)$ by the span of the classes of the $[R/Q]$ where Q is prime of height two or more (cf. the Lecture Notes of March 8 from Math 615, Fall 2004) and the fact that $G_0(r) \cong G_0(R[x])$ via the map sending $[M]$ to $[R[x] \otimes_R M]$. Since x is a prime element of $R[x]$, localizing at x does not affect the divisor class group $\square$

Discussion: inverting one entry of the matrix. We also note the following: let X be an $r \times s$ matrix indeterminates, $r, s \geq 2$. Let $x = x_{1,1}$. We can write X in block form as

$$\begin{pmatrix} x & U \\ V & W \end{pmatrix}$$

where the blocks (x), U, V, and W are 1×1, $1 \times (s-1)$, $(r-1) \times 1$, and $(r-1) \times (s-1)$, respectively. In the ring $K[X]_x$ we can perform row and column operations as follows: mutiply the first row by x^{-1}, subtract multiples of the first column from others to get the rest of the first row to be 0, and then subtract multiples of the first row from others to get the rest of the first column to be 0. This first produces

$$\begin{pmatrix} 1 & x^{-1}U \\ V & W \end{pmatrix}.$$

Multiplication on the right by

$$\begin{pmatrix} 1 & -x^{-1}U \\ 0 & 1 \end{pmatrix}$$

gives

$$\begin{pmatrix} 1 & 0 \\ V & W - x^{-1}VU \end{pmatrix}.$$

Multiplying on the left by

$$\begin{pmatrix} 1 & 0 \\ -V & 1 \end{pmatrix}$$

yields

$$\begin{pmatrix} 1 & 0 \\ 0 & W - x^{-1}VU \end{pmatrix}.$$

If we think of $W' = W - x^{-1}VU$ as new indeterminates, we have that

$$K[X]_x = K[x, U, V, W]_x = K[x, U, V, W']$$

where x, and the entries of U, V, and W' are algebraically independent over K. Now consider a submatrix X_0 of X that consists of either the first s_0 columns of X or the first r_0 rows of X. Then the expansion of $I_t(X_0)$ to $k[X]_x$ may be thought of instead as the expansion of $I_{t-1}(W'_0)$ to $K[x, U, V, W']_x$, where W'_0 is the submatrix of W' formed by the first $s_0 - 1$ columns or the first $r_0 - 1$ rows.

Theorem. *Let X be an $r \times s$ matrix of indeterminates over a field K, where $1 \leq t \leq r \leq s$. If $R = K[X]/I_t(X)$, then if $t \geq 2$, the divisor class group of R is isomorphic to the integers $\mathbb{Z}$. Let P be the prime generated by the $t-1$ size minors of any $t-1$ columns, and let Q be the prime generated by the $t-1$ size minors of any $t-1$ rows. Then P and Q are inverses in the divisor class group, and every rank one reflexive module is either isomorphic as a module with $P^{(n)}$, $n > 0$, with R, or with $Q^{(n)}$, $n > 0$. We may think of the positive integers as corresponding to the symbolic powers of P, and the negative integers as corresponding to the symbolic powers of Q.*

Proof. The fact that P and Q as specified are prime follows from our results on principal radical systems and determinantal ideals when the columns chosen are leftmost or topmost: but which $t-1$ rows or columns are chosen clearly does not affect whether the ideal is prime.

We shall first analyze the case where $t = 2$. Let $Y = (y_i)$ be an $r \times 1$ column matrix of new indeterminates and let $Z = (z_j)$ be $1 \times s$ matrix of new indeterminates as well. The product matrix $YZ = (y_i z_j)$ is $r \times s$, and so we have a surjective K-homomorphism $K[X] \twoheadrightarrow K[YZ]$ that sends $x_{i,j} \mapsto y_i z_j$. The kernel evidently contains the prime ideal $I_2(X)$, and so we have a K-algebra surjection $R \twoheadrightarrow K[y_i z_j]$. The elements $y_1 z_j$ and the ratios y_i/y_1 are algebraically independent and generate the fraction field of the latter ring, since $y_i z_j = (y_i/y_1) y_1 z_j$. The second ring therefore has Krull dimension $r + s - 1$, and the same is true for R. Therefore $R \cong K[YZ]$. Note that $K[YZ]$ has as a K-basis all monomials $y_1^{i_1} \, \ldots \, , \, y_r^{i_r} z_1^{j_1} \, \ldots \, , \, z_s^{j_s}$ such that the $i_\mu \geq 0$, $j_\nu \geq 0$ and $\sum_{\mu=1}^{r} i_\mu = \sum_{\nu=1}^{s} j_\nu$.

Note that the ideal of R generated by the entries of any one column gives the same R-module as the ideal generated by the entries of another column: for the columns numbered j and j', multiplication by $x_{i,j'}/x_{i,j}$ provides the isomorphism. This fraction does not depend on i because the 2×2 minors vanish. A similar comment applies to the rows. For definiteness, we work with the first column and the first row. Then $[P]$ and $[Q]$ are inverses because $x_{1,1}R = P \cap Q$. We shall show that $P^n = P^{(n)}$. Since $z_1^n K[Y, Z]$ is primary in $K[Y, Z]$, so is its contraction to $K[YZ]$. But the contraction is spanned by all monomials in $K[YZ]$ involving z_1^n. We can pair off the factors equal to z_1 with various y_j that must occur, and so we see that each such monomial is in P^n. It follows that P^n is a primary ideal in $K[YZ]$, and so must be equal to $P^{(n)}$. The effect of inverting $x_{1,1}$ is to produce a localized polynomial ring, whose divisor class group is trivial. The kernel of the map to this trivial group, which is all of $\mathcal{C}\ell\,(R)$, is spanned by the classes of the height one primes that meet the multiplicative system generated by x. These are simply P, and Q, and $[Q] = -[P]$. Therefore, P spans $\mathcal{C}\ell\,(R)$, which is therefore cyclic. Since the rank one reflexive modules P^n are distinct (they have minimal generating sets of mutually distinct cardinalities), the group must be infinite cyclic.

This handles the case where $t = 2$. If $t \geq 3$, the element $x = x_{1,1}$ is prime in R, by our results on principal radical systems for determinantal rings. This gives and isomorphism $\mathcal{C}\ell\,(R) \cong \mathcal{C}\ell\,(R_x)$. By the Discussion preceding the statement of the Theorem, R_x may be identified with a polynomial ring, with one variable inverted, over the ring $K[W']/I_{t-1}(W')$. By the induction hypothesis and the Proposition preceding the Discussion, the divisor class group is $\mathbb{Z}$, and has as mutually inverse generators the prime ideals P' and Q' generated by the size $t-2$ minors of the first $t-2$ columns (respectively, rows) of W'. Again, the Discussion preceding the statement of the Theorem shows that these are the expansions of P and Q (using the first $t-1$ columns or rows). $\square$

The argument given here does not show that $P^{(n)} = P^n$ and $Q^{(n)} = Q^n$ except when $t = 2$: this is true in general. One can also show that the canonical module of R is P^{s-r}: this can be reduced to the 2×2 case. We refer the reader to [W. Bruns and J. Herzog, *Cohen-Macaulay rings*, Cambridge Studies in Advanced Mathematics **39**, Cambridge University Press, Cambridge, England, 1993] §7.3 and [W. Bruns and U. Vetter, *Determinantal rings*, Lecture Notes in Mathematics **1327**, Springer, 1988], §§8 and 10 for a complete treatment.

We do, however, prove the following:

Theorem. *If $r = s$, then $K[X]/I_t(X)$ is Gorenstein.*

Proof. The canonical module is a rank one reflexive, and, with notation as in the preceding Theorem, must be isomorphic with $P^{(n)}$, $n > 0$, with R, or with $Q^{(n)}$ for $n > 0$. The K-automorphism of R induced by sending the matrix X to its transpose interchanges the roles of P and Q, and so if $P^{(n)}$ is a canonical module, so is $Q^{(n)}$. This implies that $n = 0$. □

We next aim to prove that if R is a finitely generated $\mathbb{N}$-graded algebra over $R_0 = K$, an F-finite field of characteristic $p > 0$, and Gorenstein then the R is F-regular if and only if it is F-injective, has a negative $\mathfrak{a}$-invariant, and there are forms of positive degree F_j generating an ideal primary to the homogeneous maximal ideal m such that R_{F_j} is F-regular for each j.

To this end, we need to define the $\mathfrak{a}$-invariant of a finitely generated $\mathbb{N}$-graded algebra over $R_0 = K$, a field when R is a Cohen-Macaulay ring. Let R have Krull dimension d and let $x_1, \dots, x_d$ be a homogeneous system of parameters. Then $H^d_m(R) \cong R_x / \sum_{j=1}^d R_{y_j}$, where $x = \prod_{i=1}^d x_i$ and $y_j = \prod_{i \neq j} x_i$, so that $x_j y_j = x$. This gives $H^d_m(R)$ a $\mathbb{Z}$-grading in which the degree of $(r; x_1^t, \dots, x_d^t)$ is

$$\deg(r) - t \sum_{i=1}^{d} \deg(x_i)$$

when r is homogeneous. This grading is easily checked to be indepenedent of the choice of homogeneous system of parameters. Since $H^d_m(R)$ has DCC, its nth graded piece is 0 for all sufficiently large n: otherwise, then modules $M_n = \oplus_{i \geq n} [H^d_m(R)]_n$ form an infinite descending chain of submodules of $H^d_m(R)$, a contradiction.

We define the $\mathfrak{a}$-*invariant* $\mathfrak{a}(R)$ of R to be the largest integer $a \in \mathbb{Z}$ such that $[H^d_m(R)]_a \neq 0$. A nonzero form occurring in this degree must be killed by m, and so it is in the socle of $H_d(R)$. For any homogeneous system of parameters $x_1, \dots, x_d$, the socle of $H^d_m(R)$ may be identified with the socle in $R/(x_1, \dots, x_d)R$. Hence, we can compute the $\mathfrak{a}$-invariant by considering an element of largest degree D surviving in $R/(x_1, \dots, x_d)R$: the $\mathfrak{a}$-invariant will be $D - \sum_{i=1}^d \deg(x_i)$.