Math 711: Lecture of October 31, 2005

We next observe:

Theorem. *Let R be an $\mathbb{N}$-graded finitely generated algebra over $R_0 = K$, a field of characterstic $p > 0$. Suppose that R is Cohen-Macaulay and F-injective. Then $\mathfrak{a}(R) \leq 0$.*

Proof. Suppose that R has Krull dimension d and that $x_1, \ldots, x_d$ is a homogeneous system of parameters. If $(r; x_1^t, \ldots, x_d^t)$ is a typical homogeneous element of $H^d_m(R)$, its image under F is $(r^p; x_1^{tp}, \ldots, x_d^{tp})$. This implies that for every k, F takes k-forms in $H^d_m(R)$ to pk-forms. If $u \in [H^d_m(R)]_k$ with $k > 0$ and N is such that $[H^d(R)]_n = 0$ for all $n \geq N$, we may choose $q = p^e$ with $q \geq N$. Then $F^e(u)$ has degree $qk \geq N$, and so $F^e(u) = 0$. Since F acts injectively on $H^d_m(R)$, so does F^e. It follows that $u = 0$. $\square$

Discussion: some facts about tight closure. For the next three paragraphs, we assume that all given rings are Noetherian of positive prime characteristic p. We recall that a ring that is Gorenstein and weakly F-regular (i.e., every ideal is tightly closed) is F-regular (i.e., all localizations are weakly F-regular). See the Lecture Notes from November 13 and 14 from Math 715, Fall 2002.

We recall that if R is F-finite and Gorenstein, or a finitely generated Gorenstein algebra over a field (see the remark below), then $\{P \in \mathrm{Spec}\,(R) : R_P \text{ is F-regular}\}$ is Zariski open in $\mathrm{Spec}\,(R)$. See the Lecture Notes from October 3 from Math 715, Fall 2002. Hence, its complement is closed and has a defining ideal I, which may be taken to be radical. When R is a finitely generated $\mathbb{N}$-graded over $R_0 = K$, a field, this defining ideal is homogeneous. (When the field is infinite the invariance of this ideal under automorphisms implies as usual that it is homogeneous. When the field is finite one can give a proof by extending the field.) Hence, an $\mathbb{N}$-graded algebra as above is F-regular if and only if R_m is F-regular, where m is the homogeneous maximal ideal.

Results that hold for algebras finitely generated over a field when the field is F-finite generally hold without that hypothesis. One takes a p-base Λ for the field K, and adjoins all p^e th roots for all elements of a subset Γ of the p-base that is obtained by excluding a certain finite subset of Λ. This is referred to as the Γ-*construction*: a variant of this method is used similarly to enlarge the residue field of a complete local ring in equal characteristic $p > 0$.

In the field case, one simply gets an F-finite extension field: call it L. One shows that if the finite set that is excluded is large enough, then good properties of the original K-algebra (such as being reduced, or being F-injective graded Cohen-Macaulay, or being F-regular Gorenstein) are preserved when one tensors with L. One can then work over L. For details see the Lectures of October 30 through November 7 from the Lecture Notes for Math 715, Fall 2002.

In a Gorenstein local ring, the fact that one system of parameters generates a tightly closed ideal implies that the ring is F-regular. (From the tight closure of $(x_1, \ldots, x_d)$ one is able to deduce the tight closure of $(x_1^t, \ldots, x_d^t)R$ for all t. See the Lecture Notes of

October 17 from Math 715, Fall 2002. One then gets that J is tightly closed for every m-primary ideal J, since R/J embeds in a finite direct sum of copies of $R/(x_1^t, \ldots, x_d^t)R$ for all $t \gg 0$. But every ideal I is the intersection of the ideals $I + m^n$ as n varies. Note that if I is primary to a maximal ideal m, I is tightly closed in R if and only if IR_m is tightly closed in R_m. (Note that R/I is already local, and so is equal to R_m/IR_m.) Hence, an $\mathbb{N}$-graded algebra finitely generated over $R_0 = K$, a field, is F-regular if and only if the ideal generated by one (equivalently, every) homogeneous system of parametrers is tightly closed.

Finally, in a reduced F-finite Gorenstein ring R, if R_u is F-regular then u has a power that is a test element. See the October 3 Lecture Notes from Math 715, Fall 2002. This also holds in a reduced Gorenstein ring that is finitely generated over a field K.

We are now ready to prove a useful and important criterion for a graded Gorenstein ring to be F-regular. Note that neither implication is trivial in the equivalence below. We will be able to use this result to prove that the rings $K[X/r]$ and $K[X]/I_t(X)$ are F-regular.

Theorem. *Let R be an $\mathbb{N}$-graded algebra finitely generated over $R_0 = K$, a field of characteristic $p > 0$. Let m be the homogeneous maximal ideal of R. Suppose that R is Gorenstein. Then R is F-regular if and only if the following three conditions hold:*
(1) *R is F-injective.*
(2) *There exist homogeneous elements u_j of positive degree such that every R_{u_j} is F-regular and the u_j generate an ideal primary to m.*
(3) *The $\mathfrak{a}$-invariant of R is negative.*

Proof. If R is weakly F-regular and $x^p \in I^{[p]}$, then applying Frobenius repeatedly yields that $x^q \in I^{[q]}$ for all $q = p^e$, and this implies that $x \in I^* = I$. This shows that (1) is necessary for F-regularity. It is obvious that (2) is necessary. It remains to prove (3). Note that (1) implies already that the $\mathfrak{a}$-invariant is ≤ 0. We only need to eliminate the possibility that it is 0. Suppose that $x_1, \ldots, x_d$ is a homogeneous system of parameters, all of degree h. Let u be a form of R that has degree dh. We must prove that $u \in (x_1, \ldots, x_d)$, and it suffices to show that $u \in (x_1, \ldots, x_d)^*$. Since $x_1, \ldots, x_d$ is a homogeneous system of parameters, we have that R is module-finite over $K[x_1, \ldots, x_d]$, and so u satisfies an equation of integral dependence on $K[x_1, \ldots, x_d]$. It is easy to see that there is a homogeneous equation, say $u^n + r_1 u^{n-1} + \cdots + u_0 = 0$, where the $r_i \in K[x_1, \ldots, x_d]$, for if the equation is not homogeneous we may replace r_i by its graded component in degree dhi. The fact that r_i is homogeneous of degree dhi in $K[x_1, \ldots, x_d]$ (the x_j have degree h) implies that $r_i \in (I^d)^i$, where $I = (x_1, \ldots, x_d)$. Therefore, in the ring R, u is integral over the ideal I^d, and I has d generators. By the tight closure version of the Briançon-Skoda theorem, $u \in I^* = I$. See Theorem 3.2 of the manuscript *Tight closure theory and characteristic p methods* which is available from the Web Page for Math 715, Fall 2002. This completes the proof of the necessity of condition (3).

Now assume that conditions (1), (2), and (3) hold. Condition (2) implies that every u_j has a power that is a test element, and so the test ideal is m-primary. We shall show that if $x_1, \ldots, x_d$ is any system of parameters consisting of test elements and all having the same degree, say, h, then $I = (x_1, \ldots, x_d)R$ is tightly closed. The issue is whether a representative of the generator of the socle mod $(x_1, \ldots, x_d)R$ is in the tight closure, and this element, call it u, may be taken to be homogeneous. Since the $\mathfrak{a}$-invariant of R is negative, we may assume that $\deg(u) < hd$. We shall show for sufficiently large q that u^q is in $I^{[q]}$. Condition (1) then implies that $u \in I$, a contradiction.

Since $u \in I^*$, $u^q \in (I^{[q]})^*$ for all q. Since every x_j is a test element, we have that $(x_1, \ldots, x_d)u^q \in (x_1^q, \ldots, x_d^q)$ for all q, and so $u^q \in (x_1^q, \ldots, x_d^q) :_R (x_1, \ldots, x_d)$. Since $x_1, \ldots, x_d$ is a regular sequence in R, the colon ideal is $(x_1^q, \ldots, x_d^q)R + (x_1 \cdots x_d)^{q-1}R$, and so for all q there is an equation

$$u^q = \sum_{j=1}^{d} r_j x_j^q + (x_1 \cdots x_d)^{q-1} r.$$

The equation may be assumed to be homogeneous by replacing each r_j and r by its homogeneous component in the relevant degree. In particular, if $r \neq 0$, it must have degree

$$q\deg(u) - (q-1)dh = dh - q\bigl(dh - \deg(u)\bigr).$$

Since $\deg(u) < dh$, this is negative if $q > dh/\bigl(dh - \deg(u)\bigr)$ and, in particular, if $q > dh$. For such q, $r = 0$, and this shows that $u^q \in (x_1^q, \ldots, x_d^q)R = I^{[q]}$, whence $u \in I$, a contradiction. $\square$

We now want to use this result to show that the rings $K[X/r]$ and $K[X/I_t(X)]$ are F-regular. For the latter, we have shown that each such ring is an algebra retract of a similar ring in which X is a square matrix. Therefore, we may assume that X is $r \times r$ in the proof, and so we know that both rings are Gorenstein and we may apply the Theorem.

The fact that these rings are F-injective has already been established, using that they are ASLs on a sufficiently well behaved poset. If one localizes $K[X/r]$ at any minor one gets a localization of a polynomial ring, while if one localizes $K[X]/I_t(X)$ at any $x_{i,j}$, the situation is the same as when one localizes at $x_{1,1}$. One gets a polynomial ring, localized at one variable, over a ring of the form $K[W]/I_{t-1}(W)$, where W is an $(r-1) \times (r-1)$ matrix of indeterminates. It follows by induction on t or r that this ring is F-regular. Thus, properties (1) and (2) both hold.

It remains to show the $\mathfrak{a}$-invariant is negative for each of these rings. There are many ways to do this. We make use of the fact that the $\mathfrak{a}$-invariant only depends on the Hilbert function or, equivalently, the Hilbert-Poincaré polynomial $P_R(z)$, which is defined as the formal power series $\sum_{i=0}^{\infty} \dim_K [R]_i z^i$.

If x is a nonzerodivisor in R and has degree k, the short exact sequence

$$0 \longrightarrow R(-k) \xrightarrow{x} R \longrightarrow R/xR \longrightarrow 0$$

shows that $P_{R/xR}(z) = (1 - z^h)P_R(z)$. Hence, if $x_1, \ldots, x_d$ is a homogeneous system of parameters in the Cohen-Macaulay ring R and $\deg(x_i) = k_i$, then with $A = R/(x_1, \ldots, x_d)$, we have that

$$P_A(z) = \prod_{i=1}^{d} (1 - z^{k_i}) P_R(z).$$

The highest degree of a surviving form in A is the same as the degree of $P_A(z)$, which is a polynomial, and we find that the $\mathfrak{a}$-invariant of R is the same as the degree of the Hilbert-Poincaré series viewed as a rational function of z.

Next note that the value on n of the Hilbert function of a Hodge algebra over a field is the same as the number of standard monomials of degree n. If we start with a Hodge algebra over $\mathbb{Z}$, the Hilbert function will be the same no matter what field we tensor with. It follows that the $\mathfrak{a}$-invariant of $K[X/r]$ and of $K[X]/I_t(X)$ is the same no matter what K is. Therefore, it suffices to prove that the $\mathfrak{a}$-invariant is negative in characteristic 0. We shall do this by using some classical invariant theory.